

# Correcting Limited-Magnitude Errors in the Rank-Modulation Scheme

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**Abstract**—We study error-correcting codes for permutations under the infinity norm, motivated the rank-modulation scheme for flash memories. In this scheme, a set of  $n$  flash cells are combined to create a single virtual multi-level cell. Information is stored in the permutation induced by the cell charge levels. Spike errors, which are characterized by a limited-magnitude change in cell charge levels, correspond to a low-distance change under the infinity norm.

We define codes protecting against spike errors, called *limited-magnitude rank-modulation codes* (LMRM codes), and present several constructions for these codes, some resulting in optimal codes. These codes admit simple recursive, and sometimes direct, encoding and decoding procedures.

We also provide lower and upper bounds on the maximal size of LMRM codes both in the general case, and in the case where the codes form a subgroup of the symmetric group. In the asymptotic analysis, the codes we construct out-perform the Gilbert-Varshamov-like bound estimate.

## I. INTRODUCTION

Recently (see [2]), the rank-modulation scheme has been suggested as an efficient method for storing information in flash memory devices. Let us consider  $n$  flash memory cells which we name  $1, 2, \dots, n$ . The charge level of each cell is denoted by  $c_i \in \mathbb{R}$  for all  $i \in [n]$ . In the rank-modulation scheme, the charge levels of the cells induce a permutation in the following way: The induced permutation (in vector notation) is  $[f_1, f_2, \dots, f_n]$  iff  $c_{f_1} > c_{f_2} > \dots > c_{f_n}$ .

For a measure of the corruption of a stored permutation we may use any of a variety of metrics over the symmetric group  $S_n$  (see [1]). Given a metric over  $S_n$ , defined by a distance function  $d : S_n \times S_n \rightarrow \mathbb{N} \cup \{0\}$ , an *error-correcting code* is a subset of  $S_n$  with lower-bounded distance between distinct members.

In this work we consider a limited-magnitude spike error. Suppose a permutation  $f \in S_n$  was stored by setting the charge levels of  $n$  flash memory cells to  $c_1, c_2, \dots, c_n$ . We say a single *spike error of limited-magnitude*  $L$  has occurred in the  $i$ -th cell if the corrupted charge level,  $c'_i$ , obeys  $|c_i - c'_i| \leq L$ . In general, we say spike errors of limited-magnitude  $L$  have occurred if the corrupted charge levels of all the cells,  $c'_1, c'_2, \dots, c'_n$ , obey  $\max_{i \in [n]} |c_i - c'_i| \leq L$ . We therefore find it suitable to use the  $\ell_\infty$ -metric over  $S_n$  defined by the distance function  $d_\infty(f, g) = \max_{i \in [n]} |f(i) - g(i)|$ , for all  $f, g \in S_n$ .

Since this will be the distance measure used throughout the paper, we will usually omit the  $\infty$  subscript.

**Definition 1.** A limited-magnitude rank-modulation code (LMRM-code) with parameters  $(n, M, d)$ , is a subset  $C \subseteq S_n$  of cardinality  $M$ , such that  $d_\infty(f, g) \geq d$  for all  $f, g \in C$ ,  $f \neq g$ . (We will sometimes omit the parameter  $M$ .)

## II. CODE CONSTRUCTIONS

The following is a simple construction (discovered independently and concurrently by [4]).

**Construction 1.** Given  $n, d \in \mathbb{N}$  we construct the code  $C = \{f \in S_n \mid f(i) \equiv i \pmod{d}\}$ .

**Theorem 2.** The code  $C$  from Construction 1 is an  $(n, M, d)$ -LMRM code,  $M = (\lceil n/d \rceil!)^{n \bmod d} (\lfloor n/d \rfloor!)^{d - (n \bmod d)}$ .

This construction allows a simple encoding and decoding procedures, as well as an immediate generalization to a product-like construction. Let  $\iota \in S_n$  denote the identity permutation. For convenience, given a set  $H \subseteq S_n$ , we denote  $d(H) = \min_{f, g \in H, f \neq g} d(f, g)$  and  $\bar{d}(H) = \max_{f, g \in H, f \neq g} d(f, g)$ . Finally, we recall the following notation: For  $H, K \subseteq S_n$  we denote  $H^K = \{h^k = khk^{-1} \mid h \in H, k \in K\}$ .

**Construction 2.** Let  $H$  and  $K$  be subgroups of  $S_n$  such that  $H^K = H$  and  $H \cap K = \{\iota\}$ . We construct the code  $C = H \rtimes K \cong HK = \{hk \mid h \in H, k \in K\}$ .

**Theorem 3.** The code from Construction 2 is an  $(n, M, d)$ -LMRM subgroup code with  $M = |H||K|$  and  $d \geq \min \left\{ d(H), d(K), \max \left\{ d(H) - \bar{d}(K), d(K) - \bar{d}(H) \right\} \right\}$ .

The lower bound on the distance given in Theorem 3, which we shall call the *design distance*, is often not tight.

## III. BOUNDS

Apart from the obvious analogues of the Gilbert-Varshamov bound, and the ball-packing bound, we can state a code-anticode bound. Given a set  $A \subseteq S_n$ , we denote  $\mathcal{D}(A) = \{d(f, g) \mid f, g \in A\}$ . We also denote the *inverse* of  $A$  as  $A^{-1} = \{f^{-1} \mid f \in A\}$ .

**Definition 4.** Two sets,  $A, B \subseteq S_n$  are said to be a set and an antiset if  $\mathcal{D}(A) \cap \mathcal{D}(B) = \{0\}$ .

**Theorem 5.** Let  $d : S_n \times S_n \rightarrow \mathbb{N} \cup \{0\}$  be a distance measure inducing a right-invariant metric. Let  $A, B \subseteq S_n$  be a set and an antiset. Then  $|A| \cdot |B| \leq |S_n| = n!$ .

We can apply the set-antiset method and get:

**Theorem 6.** If  $C$  is an  $(n, M, d)$ -LMRM code, then

$$M \leq \frac{n! \left( \left\lfloor \frac{d+r}{2} \right\rfloor - r \right)! \left( \left\lceil \frac{d+r}{2} \right\rceil - r \right)!}{(d!)^{\lfloor \frac{n}{d} \rfloor - 1} (d-r)! \left\lfloor \frac{d+r}{2} \right\rfloor! \left\lceil \frac{d+r}{2} \right\rceil!},$$

where  $r = n \bmod d$ .

Some of the constructions and bounds presented take on a simple asymptotic form. We will compare them with those implied by the previous constructions of [5].

**Definition 7.** Given an  $(n, M, d)$ -LMRM code, we say it has rate  $R = \frac{\log_2 M}{n}$  and normalized distance  $\delta = \frac{d}{n}$ .

We begin with the asymptotic form of Theorem 6:

**Theorem 8.** For any  $(n, M, d)$ -LMRM code,

$$R \leq \left( \delta \left\lfloor \frac{1}{\delta} \right\rfloor - \delta \right) \log_2 \left( \left\lfloor \frac{1}{\delta} \right\rfloor - 1 \right) + H_2 \left( \delta \left\lfloor \frac{1}{\delta} \right\rfloor - \delta \right) + 2 - 2\delta \left\lfloor \frac{1}{\delta} \right\rfloor + o(1)$$

where  $H_2$  is the binary entropy function.

We now state the asymptotic form of the Gilbert-Varshamov-like bound:

**Theorem 9.** For any constant  $0 < \delta \leq 1$  there exists an infinite sequence of  $(n, M, d)$ -LMRM codes with  $\frac{d}{n} \geq \delta$  and rate  $R = \frac{\log_2 M}{n}$  satisfying  $R \geq f_{GV}(\delta) + o(1)$ , where

$$f_{GV}(\delta) = \begin{cases} \log_2 \frac{1}{\delta} + 2\delta(\log_2 e - 1) - 1 & 0 < \delta \leq \frac{1}{2} \\ -2\delta \log_2 \frac{1}{\delta} + 2(1 - \delta) \log_2 e & \frac{1}{2} \leq \delta \leq 1 \end{cases}$$

The ball-packing bound has the following asymptotic form:

**Theorem 10.** For any  $(n, M, d)$ -LMRM code,

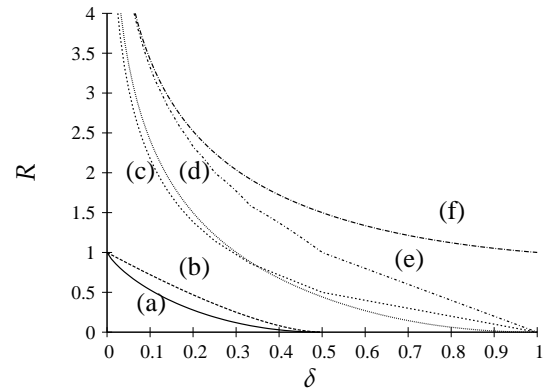
$$R \leq \delta + \log_2 \frac{1}{\delta} + o(1).$$

Finally, we analyze the asymptotics of the codes produced by Construction 1.

**Theorem 11.** For any constant  $0 < \delta \leq 1$ , Construction 1 produces codes of rate

$$R = \left( 1 - \delta \left\lfloor \frac{1}{\delta} \right\rfloor \right) \log_2 \left( \left\lfloor \frac{1}{\delta} \right\rfloor! \right) + \left( \delta + \delta \left\lfloor \frac{1}{\delta} \right\rfloor - 1 \right) \log_2 \left( \left\lfloor \frac{1}{\delta} \right\rfloor! \right).$$

All the asymptotic bounds are shown in Figure 1. Several interesting observations can be made. First, the ball-packing bound of Theorem 10 is weaker than the code-anticode bound of Theorem 8. This, however, may be due to a poor lower



**Figure 1.** (a) The Gilbert-Varshamov bound in the  $n$ -cube (b) The MRRW bound in the  $n$ -cube (c) The rate of the code from Construction 1 (d) The Gilbert-Varshamov-like bound of Theorem 9 (e) The code-anticode bound of Theorem 8 (f) The ball-packing bound of Theorem 10

bound on the size of a ball (which is not known exactly). It was conjectured in [3] that this lower bound might be improved substantially. We also note that Construction 1 produces codes which asymptotically out-perform the Gilbert-Varshamov-like bound of Theorem 9 for a wide range of  $\delta$  (with crossover at  $\delta \approx 0.34904$ ), and appear to be quite close to the bound otherwise. Again, this might be a result of a weak upper bound on the size of a ball. Finally, the codes presented by [5] are severely restricted since they are derived from binary codes in the  $n$ -cube, and as such, are bounded by the  $n$ -cube versions of the Gilbert-Varshamov bound and the MRRW bound (see, for example, [6]).

#### IV. CONCLUSION

We have studied codes for the rank modulation scheme which protect against limited-magnitude errors. We presented code constructions (some of which admit simple encoding and decoding procedures). We also explored bounds on the parameters of these codes. The strongest upper bound appears to be the code-anticode bound. For a full description of the results, the reader is referred to [7].

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