Menger's Paths with Minimum Mergings

Guangyue Han
University of Hong Kong
Pokfulam Road, Hong Kong, China
Email: ghan@hku.hk

***Abstract*—For an acyclic directed graph with multiple sources and multiple sinks, we prove that one can choose the Menger's paths between the sources and the sinks such that the number of mergings between these paths is upper bounded by a constant depending only on the min-cuts between the sources and the sinks, regardless of the size and topology of the graph. We also give bounds on the minimum number of mergings between these paths, and discuss how it depends on the min-cuts.**

I. INTRODUCTION AND NOTATION

Let $G(V, E)$ denote an acyclic directed graph, where V denotes the set of all the vertices (points) in G and E denotes the set of all the edges in G. Using these notations, the edge-connectivity version of Menger's theorem [7] states:

Theorem I.1 (Menger, 1927)**.** *For any $u, v \in V$, the maximum number of pairwise edge-disjoint directed paths from u to v in G equals the min-cut between u and v, namely the minimum number of edges in E whose deletion destroys all directed paths from u to v.*

We call any set consisting of the maximum number of pairwise edge-disjoint directed paths from u to v a set of *Menger's paths* from u and v. Apparently, for fixed $u, v \in V$, there may exist multiple sets of Menger's paths.

For m paths $\beta_1, \beta_2, \cdots, \beta_m$ in $G(V, E)$, we say these paths *merge* at $e \in E$ if

1) $e \in \cap_{i=1}^{m} \beta_i$,
2) there are at least two distinct $f, g \in E$ such that f, g are immediately ahead of e on some β_i, β_j, $j \neq i$, respectively.

Roughly speaking, condition 1 says that $\beta_1, \beta_2, \cdots, \beta_m$ *internally intersect* at e (namely, all β_i's share a common edge e), condition 2 says immediately before all β_i's internally intersect at e, at least two of them are different. We call e together with the subsequent shared edges (by all β_i's) *merged subpath* by β_i ($i = 1, 2, \cdots, m$) at e; and we often say all β_i's merge at the above-mentioned merged subpath. **In this paper we will count number of mergings without multiplicities: all the mergings at the same edge e will be counted as one merging at e.**

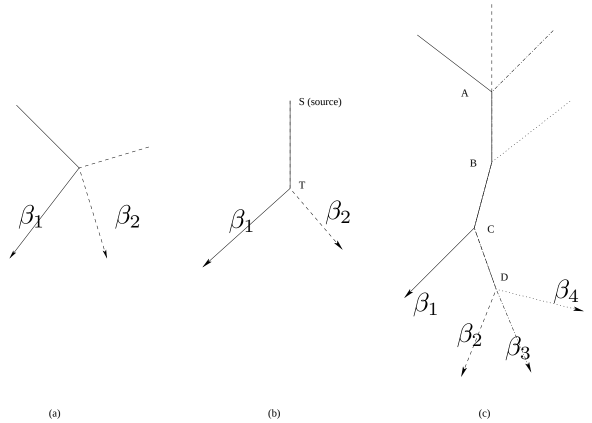

Fig. 1. examples of mergings and non-mergings

Example I.2. In Figure 1(a), paths β_1 and β_2 share some vertex, however not edges/subpaths, so β_1 and β_2 do not merge. In Figure 1(b), paths β_1 and β_2 do share edge $S \to T$, where S is a source, however condition 2 is not satisfied, therefore β_1 and β_2 do not merge, although they internally intersect at $S \to T$. In Figure 1(c), β_1 and β_2 merge at edge $A \to B$, at subpath $A \to B \to C$; β_2 and β_3 merge at edge $A \to B$, at subpath $A \to B \to C \to D$; β_1, β_2 and β_3 merge at edge $A \to B$, at subpath $A \to B \to C$; β_4 merges with β_3 at edge $B \to C$, at subpath $B \to C \to D$; there are two mergings in Figure 1(c), at edge $A \to B$, and at edge $B \to C$, respectively.

In this paper, we will consider an acyclic directed graph $G(E, V)$ with n sources and n sinks. Unless specified otherwise, we will use $S_1, S_2, \cdots, S_n$ to denote the sources and $R_1, R_2, \cdots, R_n$ to denote the sinks; c_i will be used to denote the min-cut between S_i and R_i, and $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \cdots, \alpha_{i,c_i}\}$ will be used to denote a set of Menger's paths from S_i and R_i. We will study how α_i's merge with each other; more specifically, we show that appropriately chosen Menger's paths will only merge with each other finitely many times. In particular, we deal with the case when all sources and sinks are distinct in Section II, and the case when the sources are identical and the sinks are distinct in Section III. For both of cases, we will study how the minimum merging number depends on the min-cuts.

We remark that when $n = 1$, Ford-Fulkerson algorithm [2] can find the min-cut and a set of Menger's path between S_1 and R_1 in polynomial time. The LDP (Link Disjoint Problem) asks if there are two edge-disjoint paths from S_1, S_2 to R_1, R_2, respectively. The fact that the LDP problem is NP-complete [3] suggests the intricacy of the problem when $n \geq 2$.

Notation and Convention:

For a path γ in an acyclic direct graph G, let $a(\gamma), b(\gamma)$ denote the starting point and the ending point of γ, respectively; let $\gamma[s, t]$ denote the subpath of γ with the starting point s and the ending point t. For two distinct paths γ, π in G, we say γ is *smaller* than π if there is a directed path from $b(\gamma)$ to $a(\pi)$; if γ, π and the connecting path from $b(\gamma)$ to $a(\pi)$ are subpaths of path β, we say γ is *smaller* than π on β. Note that this definition also applies to the case when paths degenerate to vertices/edges; in other words, in the definition, γ, π or the connecting path from $b(\gamma)$ to $a(\pi)$ can be vertices/edges in G, which can be viewed as degenerated paths. If $b(\gamma) = a(\pi)$, we use $\gamma \circ \pi$ to denote the path obtained by concatenating γ and π subsequently. For a set of vertices $v_1, v_2, \cdots, v_j$ in G, define $G|v_1, \cdots, v_j)$ to be subgraph of G consisting of the set of vertices, each of which is smaller than some b_j, and the set of all the edges, each of which is incident with some above-mentioned vertex.

II. MINIMUM MERGINGS $\mathcal{M}$

In this section, we consider any acyclic directed graph G with n distinct sources and n distinct sinks. Let $M(G)$ denote the minimum number of mergings over all possible Menger's path sets α_i's, $i = 1, 2, \cdots, n$, and let $\mathcal{M}(c_1, c_2, \cdots, c_n)$ denote the supremum of $M(G)$ over all possible choices of such G.

In the following, we shall prove that

Theorem II.1. *For any* $c_1, c_2, \cdots, c_n$,

$$\mathcal{M}(c_1, c_2, \cdots, c_n) < \infty,$$

and furthermore, we have

$$\mathcal{M}(c_1, c_2, \cdots, c_n) \leq \sum_{i<j} \mathcal{M}(c_i, c_j).$$

Now consider

$$\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \cdots, \alpha_{i,c_i}\},$$

a set of Menger's paths from S_i to R_i, and

$$\alpha_j = \{\alpha_{i,1}, \alpha_{i,2}, \cdots, \alpha_{i,c_j}\},$$

a set of Menger's paths from S_j to R_j. For two merged subpaths u, v by α_i and α_j (more rigorously, by some paths from α_i and α_j), we say v is *semi-reachable through* α_i by u if there is a sequence of merged subpaths $\gamma_0, \gamma_1, \cdots, \gamma_n$ by α_i and α_j such that

1) $\gamma_0 = u$, $\gamma_n = v$;
2) For each feasible k, γ_{2k+1} is smaller than γ_{2k} on some α_{j,t_k}, and $\alpha_{j,t_k}[b(\gamma_{2k+1}), a(\gamma_{2k})]$ doesn't merge with any paths from α_i;
3) For each feasible k, γ_{2k+1} is smaller than γ_{2k+2} on some α_{i,h_k}.

We say v is *regularly-semi-reachable through* α_i by u if besides the three conditions above, we further require that all h_k's in condition 3 are distinct from each other. If n is an even number, we say v is semi-reachable through α_i by u *from above*; if n is an odd number, we say v is semi-reachable through α_i by u *from below* ("above" and "below" naturally come up when G is drawn in a geometric space such that smaller paths are always higher than larger paths, as exemplified in Figure 2). It immediately follows that for three merged subpaths u, v, w by α_i, α_j, if v is semi-reachable through α_i from above by u, w is semi-reachable through α_i from above by v, then w is also semi-reachable through α_i from above by u.

Proposition II.2. *Consider Menger's path sets* α_i, α_j *and merged subpaths by* α_i, α_j*. For a merged subpath* v *semi-reachable through* α_i *by a merged subpath* u *via a sequence of merged subpaths* $\gamma_0, \gamma_1, \cdots, \gamma_n$*, if none of* γ_i*'s is semi-reachable through* α_i *by itself from above, then* v *is regularly-semi-reachable through* α_i *by* u.

Sketch of the proof: For any $k < l$ such that $h_k = h_l$ and $h_k, h_{k+1}, h_{k+2}, \cdots, h_{l-1}$ are all distinct from each other, since none of γ_i's is semi-reachable through α_i by itself from above, one checks that v is semi-reachable through α via a shorter sequence

$$\gamma_0, \cdots, \gamma_{2k+1}, \gamma_{2l+2}, \cdots, \gamma_n.$$

Continue to find such shorter immediate sequences iteratively in the similar fashion until all h_k's (corresponding to the new immediate sequence) are all distinct from each other. ■

Proposition II.3. *Consider Menger's path sets* α_i, α_j *and merged subpaths by* α_i, α_j*. If a merged subpath* u *is semi-reachable through* α_i *by itself from above via a sequence of merged subpaths* $\gamma_0, \gamma_1, \cdots, \gamma_{2m} = \gamma_0$*, then one can find a new set, still denoted by* α_i*, of* m *pairwise edge-disjoint paths from* S_i *to* R_i *such that the number of mergings between* α_j *and the new* α_i *strictly decreases.*

To see this, suppose we start with some α_i-path. When this α_i-path reaches $b(\gamma_{2k+1})$, instead of continuing on its original "trajectory", it continues on $\alpha_{j,t_k}[b(\gamma_{2k+1}), b(\gamma_{2k})]$, and then from $b(\gamma_{2k})$ it continues on the α_i-path (typically different from the original α_i-path we start with) incident with $b(\gamma_{2k})$. For instance, we can apply the above operations to the case when u is regularly-semi-reachable through α_i by itself from above; then one can reroute α_i to obtain a set of m pairwise edge-disjoint paths from S_i to R_i, by replacing $\alpha_{i,h_k}[b(\gamma_{2k+1}), R_i]$ by $\alpha_{j,t_k}[b(\gamma_{2k+1}), a(\gamma_{2k})] \circ \alpha_{i,h_{k-1}}[a(\gamma_{2k}), R_i]$ for all feasible k (here $h_0 \triangleq h_m$). Note that the above replacement "deserts" certain subpaths in the original α_i and "borrows" other subpaths from α_j to obtain a new Menger's path set α_i from S_i to R_i. We call such replacement a *rerouting* of α_i (in this case, using subpaths of α_j). After such reroutings, the number of mergings between α_i and α_j strictly decreases (however the number of mergings by all α_i's, $i = 1, 2, \cdots, n$, will probably remain the same).

The following proposition deals with the opposite direction of Proposition II.3 for the case when G has 2 distinct sources and 2 distinct sinks.

Proposition II.4. *Consider the case when there are* 2 *distinct sources and* 2 *distinct sinks in* G*. For any rerouting of* α_1

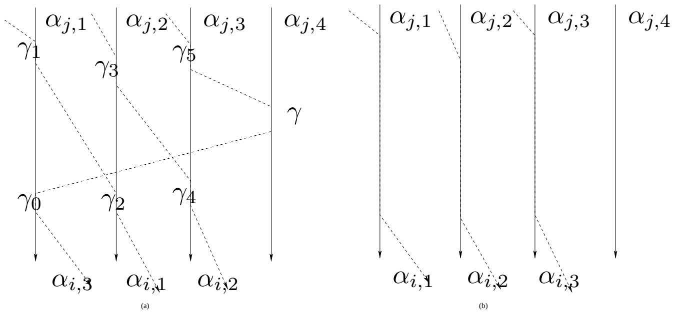

Fig. 2. an example

using α_2-subpaths, there is a merged subpath semi-reachable through α_1 by itself from above.

Proof:

Assume that subpaths $\gamma_1, \gamma_2, \cdots, \gamma_l$ are the "deserted" subpaths for a given rerouting of α_1, and these subpaths "spread" out to $\alpha_{1,1}, \alpha_{1,2}, \cdots, \alpha_{1,k}$, $k \leq l$. Without loss of generality, further assume that $\{\gamma_1, \gamma_2, \cdots, \gamma_k\}$ are the smallest such deserted subpaths on $\alpha_{1,1}, \alpha_{1,2}, \cdots, \alpha_{1,k}$, respectively. Then there are α_2-subpaths $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k$ such that all ε_i's do not merge with any α_1-paths, and for each i with $1 \leq i \leq k$ and correspondingly certain k_i with $1 \leq k_i \leq l$, $a(\varepsilon_i) = a(\gamma_i)$, $b(\varepsilon_i) = b(\gamma_{k_i})$. Surely one can find a subset $\{\hat{k}_1, \hat{k}_2, \cdots, \hat{k}_s\}$ of $\{1, 2, \cdots, k\}$ such that $b(\varepsilon_1) \in \alpha_{1,\hat{k}_1}$, $b(\varepsilon_{\hat{k}_1}) \in \alpha_{1,\hat{k}_2}$, $\cdots$, $b(\varepsilon_{\hat{k}_s}) \in \alpha_{1,1}$, which implies that there is a merged subpath (for instance, the one merged by γ_1 and $\varepsilon_{\hat{k}_s}$) semi-reachable through α_1 by itself from above.

■

Remark II.5. Consider any set of edge-disjoint paths $\beta = \{\beta_1, \beta_2, \cdots, \beta_m\}$ in G. If we add "imaginary" source S together with m disjoint edges from S to all $a(\beta_i)$'s, and add "imaginary" sink R together with m disjoint edges from all $b(\beta_i)$'s to R, we obtain a set of Menger's paths from S to R in the graph extended from G. In this section, we don't differentiate between a set of Menger's paths and a set of edge-disjoint paths for simplicity, since we can always assume the existence of such imaginary sources and sinks when they are needed.

Example II.6. In Figure 2(a), γ and γ_i ($i = 0, 1, \cdots, 5$) are merged subpaths from $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \alpha_{i,3}\}$ and $\alpha_j = \{\alpha_{j,1}, \alpha_{j,2}, \alpha_{j,3}, \alpha_{j,4}\}$. By definitions, we have

1) $\gamma_1, \gamma_3, \gamma_5$ are semi-reachable through α_i from below by γ_0,
2) γ_3, γ_5 are semi-reachable through α_i from below by γ_2,
3) γ_2, γ_4 are semi-reachable through α_i from above by γ_0,
4) γ is semi-reachable through α_i from above by $\gamma_0, \gamma_2, \gamma_4$.
5) γ_0 is semi-reachable through α_i from above by itself (via the sequence of merged subpaths $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_0$) , so are γ_2, γ_4, thus a rerouting of α_j using α_j is possible by Proposition II.3 (as shown in Figure 2(b)).

Before the proof of Theorem II.1, we shall first prove the following lemma.

Lemma II.7. *For any c_1, c_2,*

$$\mathcal{M}(c_1, c_2) \leq c_1 c_2 (c_1 + c_2)/2.$$

Proof: Consider any acyclic directed graph $G(E, V)$ with 2 distinct sources S_1, S_2 and 2 distinct sinks R_1, R_2, where the min-cut between S_i and R_i is c_i for $i = 1, 2$. Let $\alpha_1 = \{\alpha_{1,1}, \cdots, \alpha_{1,c_1}\}$ be any set of Menger's paths from S_1 to R_1, and $\alpha_2 = \{\alpha_{2,1}, \cdots, \alpha_{2,c_2}\}$ be any set of Menger's paths from S_2 to R_2. Let $V_{\mathcal{M}}$ be the set of the terminal vertices (starting and ending vertices) of all the merged subpaths by α_1 and α_2. It suffices to prove that for any c_1, c_2, if $|V_{\mathcal{M}}| \geq c_1 c_2 (c_1 + c_2) + 1$, one can always reroute α_1 using α_2, or reroute α_2 using α_1 to obtain new Menger's path sets α_1, α_2 such that the number of mergings between the new α_1, α_2 is strictly less than that between the original α_1, α_2.

Now we perform certain operations on G to obtain another graph $\hat{G}$. First we delete all the edges which do not belong to any α_1-path or α_2-path; then whenever two paths β_1, β_2 from $\alpha_1 \cup \alpha_2$ (β_1, β_2 could be both α_1-paths or α_2-paths) intersect on a vertex v, however do not share any edge incident with v (for an example, see Figure 1(a)), we "detach" β_1, β_2 at v (in other words, "split" v into two copies $v^{(1)}, v^{(2)}$ and let β_1 pass $v^{(1)}$ and let β_2 pass $v^{(2)}$); next we delete all the merged subpaths by α_1 and α_2; finally we reverse the direction of the edges which only belong to some α_2-path. Note that the above operations does not add more vertices to G; and for any path in $\hat{G}$, each edge either belongs to a α_1-path or a reversed α_2-path.

Suppose that there is a cycle in $\hat{G}$ taking the following form:

$$\gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_{2n},$$

where $b(\gamma_{2n}) = a(\gamma_1)$, γ_i is a reversed α_2-subpath for any odd i and a α_1-subpath for any even i. For any vertex w in $V_{\mathcal{M}}$, let ε_w denote the merged subpath in G corresponding to w; then one checks that in G, $\varepsilon_{a(\gamma_1)}$ is semi-reachable through α_1 by itself from above via the sequence

$$\varepsilon_{a(\gamma_1)}, \varepsilon_{a(\gamma_2)}, \cdots, \varepsilon_{a(\gamma_{2n})}, \varepsilon_{b(\gamma_{2n})},$$

which implies certain reroutings can be done to reduce the number of mergings.

Next we assume that $\hat{G}$ is acyclic. Note that in $\hat{G}$, S_1, R_2 have out-degree c_1, c_2, respectively, S_2, R_1 has in-degree c_1, c_2, respectively, and any vertex in $V_{\mathcal{M}}$ has in-degree 1 and out-degree 1. It then immediately follows that $\hat{G}$ consists of $c_1 + c_2$ pairwise vertex-disjoint paths, each of which, say γ, takes the following *regular* form:

$$\gamma = \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_n,$$

where $a(\gamma_1) = S_1$ or R_2, $b(\gamma_n) = S_2$ or R_1, the terminal points of $\gamma_2, \gamma_3, \cdots, \gamma_{n-1}$ are in $V_{\mathcal{M}}$, and each of $\gamma_1, \gamma_2, \cdots, \gamma_n$ is, alternately, either a α_1-subpath or a reversed α_2-subpath. Since $|V_{\mathcal{M}}| \geq c_1 c_2 (c_1 + c_2) + 1$, out of the $c_1 + c_2$ pairwise edge-disjoint paths, there must be at least one path, say γ, taking the regular form $\gamma = \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_n$, such that $|V_{\mathcal{M}} \cap \gamma| \geq c_1 c_2 + 1$. It then follows that there are two vertices $u, v \in V_{\mathcal{M}}$ on γ, where u corresponds to the merged subpath by α_{1,i_1} and α_{2,j_1}, and v corresponds to the merged subpath by α_{1,i_2} and α_{2,j_2}, such that $(i_1, j_1) = (i_2, j_2)$. Note that if u is larger (smaller) than v on α_{1,i_1}, then u will be also larger (smaller) than v on α_{2,j_1}, otherwise we would have a cycled path $\alpha_{1,i_1}[u,v] \circ \alpha_{1,j_1}[v,u]$ in G, which contradicts the assumption that G is acyclic. Now assume that $\gamma[u,v] = \gamma_s \circ \gamma_{s+1} \circ \cdots \circ \gamma_t$. First consider the following conditions (ignoring the parathetic words for the moment):

- u is smaller (larger) than v on α_{1,i_1},
- γ_i is a α_1-subpath (reversed α_2-subpath) for $i = s+1$,
- u is the starting (ending) vertex of the corresponding merged subpath in G, v is the starting (ending) vertex of the corresponding merged subpath in G.

Then one checks that ε_v is semi-reachable by itself from above through α_2 via the sequence $\varepsilon_v, \varepsilon_{b(\gamma_{t-1})}, \cdots, \varepsilon_{b(\gamma_s)}, \varepsilon_u, \varepsilon_v$, implying a rerouting of α_2 using α_1 to reduce the number of mergings can be done. Similar arguments can be applied to other cases when any parathetic words replace the words before them.

So in any case, if $|V_{\mathcal{M}}| \geq c_1 c_2 (c_1 + c_2) + 1$, certain reroutings can be done to strictly reduce the number of mergings. Together with the fact that each merged subpath has two terminal points, we then prove that $\mathcal{M}(c_1, c_2) \leq c_1 c_2 (c_1 + c_2)/2$, establishing the lemma.

■

We are now ready for the proof of Theorem II.1.

Proof:

With Lemma II.7 being established, to prove Theorem II.1, it suffices to prove that

$$\mathcal{M}(c_1, c_2, \cdots, c_n) \leq \mathcal{M}(c_1, c_2, \cdots, c_{n-1}) + \sum_{i<n} \mathcal{M}(c_i, c_n), \tag{1}$$

for $n = 3, 4, \cdots$, inductively.

Now suppose that for $n \leq k$, $\mathcal{M}(c_1, c_2, \cdots, c_n)$ is finite and satisfies (1) and consider the case $n = k+1$. For $i = 1, 2, \cdots, k+1$, choose a set of Menger's paths $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \cdots, \alpha_{i,c_i}\}$ between S_i and R_i, and assume $\alpha_1, \alpha_2, \cdots, \alpha_k$ are chosen such that the number of mergings among themselves is no more than $\mathcal{M}(c_1, c_2, \cdots, c_k)$. By a "new" merging, we mean a merging which is among $\alpha_1, \alpha_2, \cdots, \alpha_{k+1}$, however is not among $\alpha_1, \alpha_2, \cdots, \alpha_k$. We shall prove that if the number of new mergings between α_{k+1} and $\alpha_1, \alpha_2, \cdots, \alpha_k$ is larger than or equal to

$$\mathcal{M}(c_1, c_{k+1}) + \mathcal{M}(c_2, c_{k+1}) + \cdots + \mathcal{M}(c_k, c_{k+1}) + 1,$$

certain reroutings can be done to strictly reduce the number of mergings.

By contradiction, assume the opposite of the claim above and label all the newly merged subpaths as $\gamma_1, \gamma_2, \cdots, \gamma_l$. By the Pigeonhole principle, there exists some α_i such that α_i and α_{k+1} will have more than $\mathcal{M}(c_i, c_{k+1})$ new mergings, thus reroutings of α_i or α_{k+1} can be done. If such a rerouting is in fact a rerouting of α_{k+1} using α_i, then the number of mergings between α_{k+1} and $\alpha_1, \alpha_2, \cdots, \alpha_k$ will be strictly decreased after the rerouting. So in the following we assume that the rerouting between every α_i and α_{k+1}, if exists, is a rerouting of α_i using α_{k+1}. Then after the rerouting of α_i, the new α_i will "miss" at least

$$\mathcal{M}(c_1, c_{k+1}) + \cdots + \mathcal{M}(c_{i-1}, c_{k+1})$$

$$+\mathcal{M}(c_{i+1}, c_{k+1}) + \cdots + \mathcal{M}(c_k, c_{k+1}) + 1$$

of all the newly merged subpaths, which implies the new α_j's, $j \leq k$, will all "miss" at least one of newly merged subpaths (in other words, there is γ_{l_0} such that none of α_j's, $j \leq k$, merge with α_{k+1} at γ_{l_0}). So the number of mergings between $\alpha_1, \alpha_2, \cdots, \alpha_k$ and α_{k+1} strictly decreases after the possible reroutings of all α_i's. With this contradiction, we establish the theorem.

■

The following proposition shows that $\mathcal{M}$ is symmetric on its parameters.

Proposition II.8. *For any $c_1, c_2, \cdots, c_n$, we have*

$$\mathcal{M}(c_1, c_2, \cdots, c_n) = \mathcal{M}(c_{\delta(1)}, c_{\delta(2)}, \cdots, c_{\delta(n)}),$$

where δ is any permutation on the set of $\{1, 2, \cdots, n\}$.

The following proposition shows that $\mathcal{M}$ is an "increasing" function.

Proposition II.9. *For any $m \geq n$, $c_1 \leq c_2 \leq \cdots \leq c_n$ and $d_1 \leq d_2 \cdots \leq d_m$, if $c_i \leq d_{m-n+i}$ for $i = 1, 2, \cdots, n$, then*

$$\mathcal{M}(c_1, c_2, \cdots, c_n) \leq \mathcal{M}(d_1, d_2, \cdots, d_m).$$

Together with Proposition II.8, the following proposition shows that when $\mathcal{M}$ has two parameters, $\mathcal{M}$ is "sup-linear" in all its parameters.

Proposition II.10. *For any $c_{1,0}, c_{1,1}, c_2$, we have*

$$\mathcal{M}(c_{1,0} + c_{1,1}, c_2) \geq \mathcal{M}(c_{1,0}, c_2) + \mathcal{M}(c_{1,1}, c_2).$$

Proof:

For any $c_{1,0}, c_{1,1}$ and c_2, consider the following directed graph G with 2 sources S_1, S_2 and 2 sinks R_1, R_2 such that

1) there is a set α_1 of $c_{1,0} + c_{1,1}$ edge-disjoint paths from S_1 to R_1, here $\alpha_1 = \alpha_1^{(0)} \cup \alpha_1^{(1)}$, where $\alpha_1^{(0)}$ and $\alpha_1^{(1)}$ are mutually exclusive, consisting of $c_{1,0}$, $c_{1,1}$ edge-disjoint paths, respectively, and there is a set α_2 of c_2 edge-disjoint paths from S_2 to R_2;
2) mergings by $\alpha_1^{(0)}, \alpha_2$ and mergings by $\alpha_1^{(1)}, \alpha_2$ are sequentially isolated on α_2 in the sense that on each α_2-path, the smallest merged $\alpha_1^{(1)}$-subpath is larger than the largest merged $\alpha_1^{(0)}$-subpath;
3) the minimum number of mergings in the subgraph consisting of $\alpha_1^{(0)}$ and α_2 achieves $\mathcal{M}(c_{1,0}, c_2)$, and the minimum number of mergings in the subgraph consisting of $\alpha_1^{(1)}$ and α_2 achieves $\mathcal{M}(c_{1,1}, c_2)$.

One checks that for such graph G, the min-cut between S_1 and R_1 is $c_{1,0}+c_{1,1}$, and the min-cut between S_2 and R_2 is c_2, and

$$M(G)=\mathcal{M}(c_{1,0},c_2)+\mathcal{M}(c_{1,1},c_2),$$

which implies that

$$\mathcal{M}(c_{1,0}+c_{1,1},c_2)\geq\mathcal{M}(c_{1,0},c_2)+\mathcal{M}(c_{1,1},c_2).$$

■

Proposition II.11. *For any* $c_1,c_2,\cdots,c_n$ *and any fixed* k *with* $1\leq k\leq n$*, we have*

$$\mathcal{M}(c_1,c_2,\cdots,c_n)\geq\sum_{i\leq k,j\geq k+1}\mathcal{M}(c_i,c_j).$$

Proof:

For any $c_1,c_2,\cdots,c_n$, consider the following directed graph G with n sources $S_1,S_2,\cdots,S_n$ and n sinks $R_1,R_2,\cdots,R_n$ such that for any fixed k with $1\leq k\leq n$,

1) there is a set α_i of c_i edge-disjoint paths from S_i to R_i for each i;
2) all α_i's, $i\leq k$, do not merge with each other, and all α_j's, $j\geq k+1$, do not merge with each other;
3) for any i with $i\leq k$, mergings by α_i and all α_j's, $j\geq k+1$, are sequentially isolated on α_i in the sense that on each α_i-path, for any $j_1<j_2$ with $j_1,j_2\geq k+1$, the smallest merged α_{j_2}-subpath is larger than the largest merged α_{j_1}-subpath. Similarly for any j with $j\geq k+1$, mergings by α_j and all α_i's, $i\leq k$, are sequentially isolated on α_j.
4) the minimum number of mergings in the subgraph consisting of any α_i with $i\leq k$ and any α_j with $j\geq k+1$ achieves $\mathcal{M}(c_i,c_j)$.

One checks that for such graph G, the min-cut between S_i and R_i is c_i, and

$$M(G)=\sum_{i\leq k,j\geq k+1}\mathcal{M}(c_i,c_j),$$

which implies that

$$\mathcal{M}(c_1,c_2,\cdots,c_n)\geq\sum_{i\leq k,j\geq k+1}\mathcal{M}(c_i,c_j).$$

■

The following proposition gives an upper bound on $\mathcal{M}(m,n)$ using $\mathcal{M}(m_1,n_1)$'s, where $m_1\leq m$, $n_1\leq n$.

Proposition II.12. *For any* $m\leq n$*, we have*

$$\mathcal{M}(m,n)\leq U(m,n)+V(m,n)+m-2,$$

where

$$U(m,n)=\sum_{j=1}^{m-1}\left(\mathcal{M}(j,m-1)+1+\mathcal{M}(m-j,n)\right)+\mathcal{M}(m,m-1)+1,$$

and

$$V(m,n)=\mathcal{M}(m,n-1)+\sum_{j=1}^{m-1}\left(\mathcal{M}(j,n)+1+\mathcal{M}(m-j,n)\right)-\mathcal{M}(1,n).$$

Proof:

Proof of this proposition is omitted due to space limit. For interested reader, we refer to http://arxiv.org/abs/0805.4059

■

Remark II.13. Define $w_i=\sum_{j=1}^{i}(\mathcal{M}(j,m-1)+1)$. Note that Proposition II.12 is still true if $U(m,n)$ is replaced by mw_m, which produces an alternative upper bound on $\mathcal{M}(m,n)$. One can obtain the proof of this by replacing $U(m,n)$ in the first and second paragraphs in the proof of Proposition II.12 with mw_m and replacing the third paragraph in the proof of Proposition II.12 with the following paragraph.

Now assume that we find $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ such that

$$\mathbb{S}\overset{\triangle}{=}G|b(\varepsilon_1),\cdots,b(\varepsilon_n))$$

has no less than $V(m,n)$ mergings and $\mathbb{R}=G\backslash\mathbb{S}$ has no less than mw_m mergings. By the Pigeonhole principle, there must be at least one ϕ_j such that ϕ_j merge with ψ for no less than w_m times. Without loss of generality, assume that ϕ_{j_1} merges with ψ subsequently at $\eta_1^{(1)},\eta_2^{(1)},\cdots,\eta_{l_1}^{(1)}$, here $l_1\geq w_m$. Now within $\mathbb{R}|b(\eta_{w_1}^{(1)}))$, unless ϕ_{j_1} merges with no less than m ψ-paths, there exists $j_2\neq j_1$ such that a merged ϕ_{j_2}-subpath, say $\gamma^{(1)}$, is immediately ahead of certain merged ϕ_{j_1}-subpath, say $\eta_{l_1^*}^{(1)}$. So $\gamma^{(1)}$ and any merged subpath larger than $\gamma^{(1)}$ on ϕ_{j_2} is semi-reachbable through ϕ by $\eta_{l_1^*}^{(1)}$. Now continue the argument inductively and suppose we have already obtained $j_1,j_2,\cdots,j_{k+1}$. Then within $\mathbb{R}|b(\eta_{w_{k+1}}^{(1)}))\backslash\mathbb{R}|b(\eta_{w_k}^{(1)}))$, assume that $\phi_{j_1},\phi_{j_2},\cdots,\phi_{j_{k+1}}$ merge with ψ at $\eta_1^{(k+1)},\eta_2^{(k+1)},\cdots,\eta_{l_{k+1}}^{(k+1)}$, here obviously $l_{k+1}\geq w_{k+1}-w_k$. Unless $\phi_{j_1},\phi_{j_2},\cdots,\phi_{j_{k+1}}$ merge with no less than m ψ-paths within $\mathbb{R}|b(\eta_{w_{k+1}}^{(1)}))\backslash\mathbb{R}|b(\eta_{w_k}^{(1)}))$, there exists $j_{k+2}\neq j_1,j_2,\cdots,j_{k+1}$ such that a merged $\phi_{j_{k+2}}$-subpath, say $\gamma^{(k+1)}$, is immediately ahead of some $\eta_{l_{k+1}^*}^{(k+1)}$. Thus $\gamma^{(k+1)}$ and any merged subpaths larger than $\gamma^{(k+1)}$ on $\phi_{j_{k+2}}$ are semi-reachable through ϕ by $\eta_{l_{k+1}^*}^{(k+1)}$, and thus by some $\eta_i^{(1)}$. Eventually one can show that within $\mathbb{R}|b(\eta_{w_m}^{(1)}))\backslash\mathbb{R}|b(\eta_{w_{m-1}}^{(1)}))$, all merged non-$\phi_{j_1}$-subpaths are semi-reachable through ϕ by some $\eta_i^{(1)}$. Since

$$|\mathbb{R}|b(\eta_{w_m}^{(1)}))\backslash\mathbb{R}|b(\eta_{w_{m-1}}^{(1)}))|_{\mathcal{M}}\geq\mathcal{M}(m-1,m)+1,$$

all merged subpaths within $\mathbb{R}|b(\eta_{w_m}^{(1)}))\backslash\mathbb{R}|b(\eta_{w_{m-1}}^{(1)}))$ spread out to no less than m ψ-paths, which implies that within G all the merged subpaths semi-reachable through ϕ by $\eta_1^{(1)},\eta_2^{(1)}$, $\cdots$, or $\eta_{l_1}^{(1)}$ will spread out to no less than m ψ-paths.

Example II.14. It was first shown in [8] that $\mathcal{M}(1,n)=n$. To see this, consider any acyclic directed graph $G(E,V)$ with 2 distinct sources S_1,S_2 and 2 distinct sinks R_1,R_2, where the min-cut between S_i and R_i is denoted by c_i; here $c_1=1$ and $c_2=n$. Pick a set of Menger's path $\alpha_i=\{\alpha_{i,1},\alpha_{i,2},\cdots,\alpha_{i,c_i}\}$ from S_i to R_i. If $\alpha_{1,1}$ merges with some α_2-path, say $\alpha_{2,j}$, at least twice, say at e and f. Then we can replace $\alpha_{1,1}[a(e),a(f)]$, the subpath of $\alpha_{1,1}$ starting from $a(e)$ to $a(f)$, by $\alpha_{2,j}[a(e),a(f)]$, the subpath of $\alpha_{2,j}$

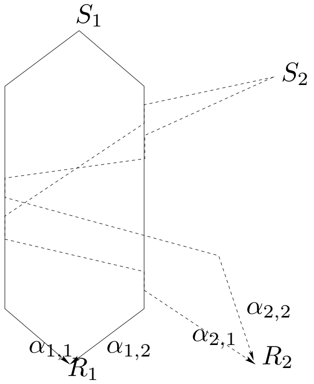

Fig. 3. an example achieving $\mathcal{M}(2,2)$

starting from $a(e)$ to $a(f)$. After this rerouting, the new $\alpha_{1,1}$ has fewer mergings with α_2. This shows that

$$\mathcal{M}(1,n) \leq n,$$

since $\alpha_{1,1}$ can be chosen to merge with each α_2-path for at most once. For the other direction, by Proposition II.10, we have

$$\mathcal{M}(1,n) \geq \sum_{i=1}^{n} \mathcal{M}(1,1) = n,$$

the last equality follows from the simple fact that $\mathcal{M}(1,1) = 1$.

Remark II.15. Note that Example II.14 together with the inductive argument in the proof of Proposition II.12 gives an alternative proof of that $\mathcal{M}(c_1, c_2)$ is finite.

Example II.16. Consider an acyclic directed graph G with 2 sources S_1, S_2 and 2 sinks R_1, R_2, where the min-cut between S_i to R_i is 2. Let $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}\}$ be a set of Mengers' paths from S_i to R_i. If any α_2-path, say $\alpha_{2,i}$, merges with some α_1-path, say $\alpha_{1,j}$, twice at two merged subpaths γ_1, γ_2, where γ_1 is immediately ahead of γ_2 on $\alpha_{1,j}$ (or $\alpha_{2,i}$), as shown in the proof of Example II.14, one can reroute $\alpha_{2,i}$ (or $\alpha_{1,j}$) to reduce the number of mergings. So we can assume that path $\alpha_{1,j}$ $(j = 1, 2)$ can be assumed to merge with paths $\alpha_{2,1}, \alpha_{2,2}$ alternately, and similarly path $\alpha_{2,j}$ $(j = 1, 2)$ can be also assumed to merge with paths $\alpha_{1,1}, \alpha_{1,2}$ alternately. This allows us to be able to exhaustively list all the possible patterns of G, where there are no possible reroutings. With the graph depicted by Figure 3, we conclude that $\mathcal{M}(2,2) = 5$. Applying Theorem II.1, we have

$$\mathcal{M}^*(\underbrace{2, 2, \cdots, 2}_{n}) \leq \frac{5n(n-1)}{2}.$$

Remark II.17. For an acyclic directed graph $G(V, E)$, the vertex-connectivity version of Menger's theorem [7] states:

For any $u, v \in V$, with no edge from u to v, the maximum number of pairwise vertex-disjoint directed paths from u to v in G equals the minimum vertex cut between u and v, namely the minimum number of vertices in $E\backslash\{u, v\}$ whose deletion destroys all directed paths from u to v.

In this remark, we redefine Menger's paths and merging: we call any set consisting of the maximum number of pairwise vertex-disjoint directed paths from u to v a set of *Menger's paths* from u and v; and for m paths $\beta_1, \beta_2, \cdots, \beta_m$ in $G(V, E)$, we say these paths *merge* at $e \in V$ (here E in the original definition is replaced by V) if

1) $e \in \cap_{i=1}^{m} \beta_i$,
2) there are at least two distinct $f, g \in E$ such that f, g are immediately ahead of e on some β_i, β_j, respectively.

And naturally we can also redefine $\mathcal{M}$ with the above redefined Menger's paths and merging. Then using a parallel argument, one can show that Theorem II.1 still hold true for redefined $\mathcal{M}$.

III. MINIMUM MERGINGS $\mathcal{M}^*$

In this section, we consider any acyclic directed graph G with one source and n distinct sinks. Let $M^*(G)$ denote the minimum number of mergings over all possible Menger's path sets α_i's, $i = 1, 2, \cdots, n$, and let $\mathcal{M}^*(c_1, c_2, \cdots, c_n)$ denote the supremum of $M^*(G)$ over all possible choices of such G.

We also have the following "finiteness" theorem for $\mathcal{M}^*$:

Theorem III.1. *For any $c_1, c_2, \cdots, c_n$,*

$$\mathcal{M}^*(c_1, c_2, \cdots, c_n) < \infty,$$

and furthermore, we have

$$\mathcal{M}^*(c_1, c_2, \cdots, c_n) \leq \sum_{i<j} \mathcal{M}^*(c_i, c_j).$$

Proof:

As illustrated in Remark II.5, we extend G to $\hat{G}$ by first adding n imaginary sources $S_1, S_2, \cdots, S_n$, and then adding c_i disjoint edges from S_i to S for each feasible i. For any such G and $\hat{G}$, one checks that the original Menger's paths (from S to each R_i for all i) merge with each other fewer times than the extended Menger's paths (from S_i to R_i for all i), which implies that

$$\mathcal{M}^*(c_1, c_2, \cdots, c_n) \leq \mathcal{M}(c_1, c_2, \cdots, c_n).$$

The finiteness result then immediately follows from Theorem II.1. As for the inequality, exactly the same argument of Theorem II.1 applies to $\mathcal{M}^*$, thus we have for any $c_1, c_2, \cdots, c_{n+1}$

$$\mathcal{M}^*(c_1, c_2, \cdots, c_n) \leq \mathcal{M}^*(c_1, c_2, \cdots, c_{n-1}) + \sum_{j<n} \mathcal{M}^*(c_j, c_n),$$

which implies the inequality. ■

Remark III.2. The same techniques as in the proof above, together with Theorem II.1, show that appropriately chosen Menger's paths merge with each other only finitely many times, if only some of the sources and/or some of the sinks are identical.

Remark III.3. Theorem II.1 and Theorem III.1 do not hold for cyclic directed graphs. As shown in Figure 4, for an arbitrary n, $\alpha_{2,1}$ merges with $\alpha_{1,2}$ at $\gamma_1, \gamma_2, \cdots, \gamma_{n-1}, \gamma_n$ subsequently

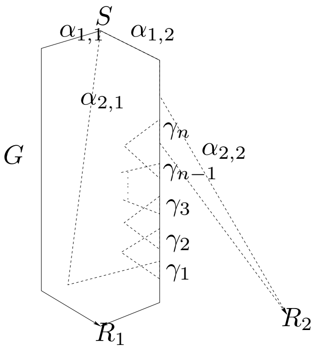

Fig. 4. an counterexample

from the bottom to the top. One checks that α_1 and α_2 has n mergings, and there is no way to reroute α_1 or α_2 to decrease the number of mergings.

Similar to $\mathcal{M}$, $\mathcal{M}^*$ is a symmetric and "increasing" function.

Proposition III.4. *$\mathcal{M}^*$ is symmetric on its parameters. More specifically,*

$$\mathcal{M}^*(c_1, c_2, \cdots, c_n) = \mathcal{M}^*(c_{\delta(1)}, c_{\delta(2)}, \cdots, c_{\delta(n)}),$$

where δ is any permuation on the set of $\{1, 2, \cdots, n\}$.

Proposition III.5. *For $m \geq n$, $c_1 \leq c_2 \leq \cdots \leq c_n$, and $d_1 \leq d_2 \cdots \leq d_m$, if $c_i \leq d_{m-n+i}$ for $i = 1, 2, \cdots, n$, then*

$$\mathcal{M}^*(c_1, c_2, \cdots, c_n) \leq \mathcal{M}^*(d_1, d_2, \cdots, d_m).$$

Proposition III.6. *For $c_1 \leq c_2 \leq \cdots \leq c_n$, if $c_1 + c_2 + \cdots + c_{n-1} \leq c_n$, then*

$$\mathcal{M}^*(c_1, c_2, \cdots, c_n) = \mathcal{M}^*(c_1, c_2, \cdots, c_{n-1}, c_1+c_2+\cdots+c_{n-1}).$$

Proof: Given any acyclic directed graph G with one source S and n sinks $R_1, R_2, \cdots, R_n$, where the min-cut between S and R_i is c_i, pick a set of Menger's paths $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \cdots, \alpha_{i,c_i}\}$ from S to R_i for all feasible i. If any path from α_n, say β, does not share subpath starting from S with any other paths and first merges with some path η at merged subpath γ, then one can reroute all such η (merging with β at γ) by replacing $\eta[S, b(\gamma)]$ by $\beta[S, b(\gamma)]$ to reduce the merging number. Note that such possible reroutings can be done to all the paths from α_n. As a result of such possible reroutings, at least $c_n - (c_1 + c_2 + \cdots + c_{n-1})$ paths from α_n will not merge with any paths from $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$, which implies

$$\mathcal{M}^*(c_1, c_2, \cdots, c_n) \leq \mathcal{M}^*(c_1, c_2, \cdots, c_{n-1}, c_1+c_2+\cdots+c_{n-1}).$$

The other direction is obvious from Proposition III.5. The proposition then immediately follows. ■

Proposition III.7. *For $c_1 = 1 \leq c_2 \leq \cdots \leq c_n$, we have*

$$\mathcal{M}^*(c_1, c_2, \cdots, c_n) = \mathcal{M}^*(c_2, \cdots, c_{n-1}, c_n).$$

Proof: Given any acyclic directed graph G with one source S and n sinks $R_1, R_2, \cdots, R_n$, where the min-cut between S and R_i is c_i, choose Menger's paths $\alpha_2, \alpha_3, \cdots, \alpha_n$ such that the number of mergings among them is less than $\mathcal{M}^*(c_2, \cdots, c_{n-1}, c_n)$. If $\alpha_{1,1}$ does not merge with any paths from $\alpha_2, \alpha_3, \cdots, \alpha_n$, then the number of mergings in G among all α_i's is less than $\mathcal{M}^*(c_2, \cdots, c_{n-1}, c_n)$; if $\alpha_{1,1}$ does merge with other paths and it last merges with, say $\alpha_{i,j}$, at γ, then we can reroute $\alpha_{1,1}$ by replacing $\alpha_{1,1}[S, a(\gamma)]$ by $\alpha_{i,j}[S, a(\gamma)]$. With rerouted $\alpha_{1,1}$, the number of mergings in G among all α_i's is still less than $\mathcal{M}^*(c_2, \cdots, c_{n-1}, c_n)$, which implies

$$\mathcal{M}^*(c_1, c_2, \cdots, c_n) \leq \mathcal{M}^*(c_2, \cdots, c_{n-1}, c_n).$$

The other direction is obvious from Proposition III.5. The Proposition then immediately follows. ■

Remark III.8. Now we can see that in terms of the dependence on the parameters, the behaviors of $\mathcal{M}$ and $\mathcal{M}^*$ can be very different. For instance,

- from Example II.14, we have $\mathcal{M}(1, 2) = 2 > 1 = \mathcal{M}(1, 1)$, which implies $\mathcal{M}$ does not satisfy the equality in Proposition III.6;
- through Proposition III.6, we see that

  $$\mathcal{M}^*(2c, c) = \mathcal{M}^*(c, c) \leq \mathcal{M}^*(c, c) + \mathcal{M}^*(c, c),$$

  and strict inequality in the above expression holds as long as $\mathcal{M}^*(c, c) > 0$, thus $\mathcal{M}^*$ does not satisfy the inequality in Proposition II.10; namely, not like $\mathcal{M}$, $\mathcal{M}^*$ is not suplinear in its parameters;
- Proposition III.7 implies that $\mathcal{M}^*(1, n) = 0$, while from Example II.14, we have $\mathcal{M}(1, n) = n$, which implies $\mathcal{M}$ does not satisfy the equality in Proposition III.7.

The following proposition reveals a relationship between $\mathcal{M}$ and $\mathcal{M}^*$.

Proposition III.9. *For any n, we have*

$$\mathcal{M}^*(n + 1, n + 1) \leq \mathcal{M}(n, n).$$

Proof: Consider the case when G has one source S and two sinks R_1, R_2, and the min-cut between the source S and every sink is equal to $n + 1$. For each sink R_i, pick a set of Menger's paths $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \cdots, \alpha_{i,n+1}\}$. By Proposition III.5, we can assume every α_1-path merges with certain α_2-path and vice versa. As shown in the proof of Proposition III.6, we can further assume $\alpha_{1,i}$ shares subpath starting from S with $\alpha_{2,i}$, $i = 1, 2, \cdots, n + 1$, after possible reroutings. Now, if every α_1-path merges with some α_2-path, for instance, $\alpha_{1,i}$ first merges with $\alpha_{2,\delta(i)}$ at merged subpath γ_i, here δ denotes certain mapping from $\{1, 2, \cdots, n + 1\}$ to $\{1, 2, \cdots, n + 1\}$, then there exists m ($m \leq n + 1$) such that $\delta^m(1) = 1$. We can further choose m to be the smallest such "period". In this case certain reroutings of α_2 can be done by replacing $\alpha_{2,\delta^j(1)}[S, b(\gamma_{\delta^{j-1}(1)})]$ by $\alpha_{1,\delta^{j-1}(1)}[S, b(\gamma_{\delta^{j-1}(1)})]$, $j = 1, \cdots, m$ (here $\delta^0(1) \overset{\triangle}{=} 1$), to reduce the merging number. So, without loss of generality, we can assume, after further possible reroutings, $\alpha_{1,n+1}$ does not merge with any other paths, and $\alpha_{2,1}$ doesn't merge with any other paths either by similar argument; in other words, all mergings are by paths $\alpha_{1,1}, \alpha_{1,2}, \cdots, \alpha_{1,n}$ and paths $\alpha_{2,2}, \alpha_{2,3}, \cdots, \alpha_{2,n+1}$, which establishes the theorem. ■

Proposition III.10. *For any n, we have*

$$\mathcal{M}^*(\underbrace{2, 2, \cdots, 2}_{n}) = \mathcal{M}^*(\underbrace{2, \cdots, 2, 2}_{n-1}) + 1.$$

Proof: Given any acyclic directed graph G with one source S and n sinks $R_1, R_2, \cdots, R_n$, where the min-cut between S and R_i is 2, pick a set of Menger's paths $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}\}$ from S to R_i for all feasible i. Again by a new merging, we mean a merging among $\alpha_1, \alpha_2, \cdots, \alpha_n$, however not among $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$. Assume that $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ are chosen such that the mergings among themselves is no more than $\mathcal{M}^*(\underbrace{2, 2, \cdots, 2}_{n-1})$, we shall prove that whenever α_n newly merges with $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ more than 2 times, one can always reroute certain paths to decrease the total number of mergings within $\alpha_1, \alpha_2, \cdots, \alpha_n$. Apparently this will be sufficient to imply:

$$\mathcal{M}^*(\underbrace{2, 2, \cdots, 2}_{n}) \leq \mathcal{M}^*(\underbrace{2, \cdots, 2, 2}_{n-1}) + 1.$$

In the following, for any j, if we use p to refer to one of the two paths in α_j, we will use $\bar{p}$ to refer to the other path in α_j. Consider the following two scenarios:

1) for two certain Menger's paths p, q, p merges with q and $\bar{p}$ merges with $\bar{q}$;
2) for a Menger's path $p \in \alpha_n$ which newly merges with $q_1, q_2, \cdots, q_l$ at subpath γ (here we have listed all the paths merging with p at γ), p shares a subpath with every q_j before the new merging.

For scenario 1, suppose p merges with q at γ, and $\bar{p}$ merges with $\bar{q}$ at ε. Then one can always reroute $p[S, a(\gamma)]$ using $q[S, a(\gamma)]$, reroute $\bar{p}[S, a(\varepsilon)]$ using $\bar{q}[S, a(\varepsilon)]$; or alternatively reroute $q[S, a(\gamma)]$ using $p[S, a(\gamma)]$, reroute $\bar{q}[S, a(\varepsilon)]$ using $\bar{p}[S, a(\varepsilon)]$. So in the following we assume that scenario 1 never occurs.

For scenario 2, suppose that before p newly merges with $q_1, q_2, \cdots, q_l$ at γ, p shares a subpath ε_j with every q_j. We can assume $\bar{p}$ merges with every $q_j[b(\varepsilon_j), a(\gamma)]$, otherwise one can reroute $p[b(\psi_j), a(\phi)]$ using $q_j[b(\psi_j), a(\phi)]$ (and thus the new merging at γ disappear); we can also assume for some path i, $\bar{q}_i$ merges with $p[b(\varepsilon_i), a(\gamma)]$, otherwise one can reroute every $q_j[b(\varepsilon_j), a(\gamma)]$ using $p[b(\varepsilon_j), a(\gamma)]$ and consequently all paths $q_1, q_2, \cdots, q_l$ can be rerouted (and thus the new merging at γ disappear). But if for some path i, $\bar{q}_i$ merges with $p[b(\varepsilon_i), a(\gamma)]$, scenario 1 occurs: p merges with q_i, and $\bar{p}$ merges with $\bar{q}_i$. So in the following we assume scenario 2 does not occur either, i.e., there is always some q_i such that before the new merging, p does not internally intersect with q_i.

We say p newly merges with q_i *essentially* at γ if

1) before the new merging, p does not internally intersect (again meaning share subpath) with q_i,
2) $\bar{p}$ internally intersects with $q_j[S, a(\gamma)]$,
3) $\bar{q}_i$ internally intersects with $p[S, a(\gamma)]$.

One checks that if p newly merges with some q_i non-essentially at γ, then either $p[S, a(\gamma)]$ or $q_i[S, a(\gamma)]$ can be rerouted. Furthermore if p newly merges with q_i essentially at γ, and $\bar{p}$ last merges with $q_i[S, a(\gamma)]$ at ε, then one can reroute $\bar{p}$ by replacing $\bar{p}[S, a(\varepsilon)]$ by $\bar{q}_i[S, a(\varepsilon)]$, so the new $\bar{p}$ shares subpath $\bar{q}_i[S, b(\varepsilon)]$ staring from S; in other words, after possible reroutings, we can further assume that $\bar{p}$ shares certain subpath with q_i starting from S.

Now suppose $p \in \alpha_n$ newly merges twice at γ_1, γ_2. For $i = 1, 2$, among all the Menger's paths merging with p at γ_i, let q_i denote an arbitrarily chosen path such that p newly merges with q_i at γ_i essentially (note that $q_1 \neq q_2$ since both of them merge with p essentially). If $\bar{q}_2$ merges with $p[b(\gamma_1), a(\gamma_2)]$ at subpath ε_1, since $\bar{q}_2$ does not merge with $\bar{p}$ (scenario 1 does not occur), one can reroute $p[S, a(\varepsilon_1)]$ using $\bar{q}_2[S, a(\varepsilon_1)]$ (then the new merging at γ_1 would disappear). Consider the case when $\bar{q}_2$ does not merge with $p[b(\gamma_1), a(\gamma_2)]$. If $\bar{q}_2$ does not merge with $q_1[S, a(\gamma_1)]$ either, one can reroute $q_2[S, a(\gamma_2)]$ using $q_1[S, a(\gamma_1)] \circ p[a(\gamma_1), a(\gamma_2)]$. Now consider the case when $\bar{q}_2$ merges with $q_1[S, a(\gamma_1)]$ and suppose $\bar{q}_2$ last merges with $q_1[S, a(\gamma_1)]$ at ε_2. If $\bar{p}$ does not merge with $q_1[b(\varepsilon_2), a(\gamma_1)]$, since $\bar{q}_2$ won't merge with $\bar{p}$, $p[S, a(\gamma_1)]$ can be rerouted using $\bar{q}_2[S, b(\varepsilon_2)] \circ q_1[b(\varepsilon_2), a(\gamma_1)]$ (then the new merging at γ_1 would disappear). Now consider the case when $\bar{p}$ does merge with $q_1[b(\varepsilon_2), a(\gamma_1)]$ at subpath ε_3. But in this case, one can reroute $q_2[S, a(\gamma_2)]$ using $\bar{p}[S, a(\varepsilon_3)] \circ q_1[a(\varepsilon_3), a(\gamma_1)] \circ p[a(\gamma_1), a(\gamma_2)]$. Apply the arguments above to arbitrarily chosen pair q_1, q_2 essentially merging with p, together with the fact that non-essential merging will disappear after appropriate reroutings, we conclude that ultimately certain reroutings to reduce the number of mergings are always possible when $p \in \alpha_n$ newly merges twice.

Now suppose $p \in \alpha_n$ newly merges at γ_1, and $\bar{p} \in \alpha_n$ newly merges at γ_2. Let q_1 denote an arbitrarily chosen path, among all the paths merging with p at γ_1, such that p newly merges with q_1 at γ_1 essentially; let q_2 denote an arbitrarily chosen path, among all the paths merging with $\bar{p}$ at γ_2, such that $\bar{p}$ newly merges with q_2 at γ_2 essentially (again one checks that $q_1 \neq q_2$ since they essentially merge with $p, \bar{p}$, respectively). Apparently q_1, q_2 must merge with each other, otherwise one can reroute $p[S, a(\gamma_1)]$ using $q_1[S, a(\gamma_1)]$ and reroute $\bar{p}[S, a(\gamma_2)]$ using $q_2[S, a(\gamma_2)]$ (then the two new mergings would disappear). Suppose q_1 and q_2 last merge at ε_1. We claim that $\bar{p}$ must merge with $q_1[b(\varepsilon_1), a(\gamma_1)]$, otherwise one can reroute $p[S, a(\gamma_1)]$ using $q_2[S, b(\varepsilon_1)] \circ q_1[b(\varepsilon_1), a(\gamma_1)]$ (p shares subpath with q_2 from S and does not merge with q_1 before γ_1). Furthermore $\bar{p}$ must merge with $q_1[b(\varepsilon_1), a(\gamma_1)]$ at least once before $a(\gamma_2)$ (in other words, $\bar{p}[S, a(\gamma_2)]$ must merge with $q_1[b(\varepsilon_1), a(\gamma_1)]$), since otherwise, say $\bar{p}[b(\gamma_2), R_n]$ merges with $q_1[b(\varepsilon_1), a(\gamma_1)]$ at ε_2, then one can reroute $\bar{p}[S, a(\varepsilon_2)]$ with $q_1[S, a(\varepsilon_2)]$ (thus the new merging at γ_2 would disappear). Similarly $p[S, a(\gamma_1)]$ must merge with $q_2[b(\varepsilon_1), a(\gamma_2)]$. Now suppose $\bar{p}[S, a(\gamma_2)]$ first merges with $q_1[b(\varepsilon_1), a(\gamma_1)]$ at subpath ε_2. Since scenario 1 does not occur, $\bar{q}_1$ won't merge with p, therefore it must share certain subpath with p staring from S (here we remind the reader that p newly merges with q_1 essentially, so $\bar{q}_1$ will either merge with or share certain subpath with p from S). Similarly suppose $p[S, a(\gamma_1)]$ first merges with $q_2[b(\varepsilon_1), a(\gamma_2)]$ at ε_3, then $\bar{q}_2$ must share certain subpath with $\bar{p}$ staring from S. Now since scenario 1 does not occur, either $\bar{q}_2$ won't merge with $q_1[b(\varepsilon_1), a(\varepsilon_2)]$ or $\bar{q}_1$ won't merge with $q_2[b(\varepsilon_1), a(\varepsilon_3)]$. If $\bar{q}_2$ does not merge with $q_1[b(\varepsilon_1), a(\varepsilon_2)]$, then one can reroute $q_2[b(\varepsilon_1), a(\gamma_2)]$ with $q_1[b(\varepsilon_1), a(\varepsilon_2)] \circ \bar{p}[a(\varepsilon_2), a(\gamma_2)]$; if $\bar{q}_1$ does not merge with $q_2[b(\varepsilon_1), a(\varepsilon_3)]$, then one can reroute $q_1[b(\varepsilon_1), a(\gamma_1)]$ with $q_2[b(\varepsilon_1), a(\varepsilon_3)] \circ p[a(\varepsilon_3), a(\gamma_1)]$. Apply the arguments above to arbitrarily chosen pair q_1, q_2 essentially merging with p, together with the fact that non-essential merging will disappear after appropriate reroutings, we conclude that ultimately certain reroutings to reduce the number of mergings are always possible when when $p \in \alpha_n$ newly merges and $\bar{p} \in \alpha_n$ newly merges.

For the other direction, assume that the subgraph consisting of $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ achieves $\mathcal{M}^*(\underbrace{2, 2, \cdots, 2}_{n-1})$, we add α_n such that for $i = 1, 2$, $\alpha_{n,i}$ share subpath with $\alpha_{n-1,i}$, α_n only merges with α_{n-1} once, say $\alpha_{n,1}$ merges with $\alpha_{n-1,2}$ at γ, where γ is a largest merged subpath. One checks the graph consisting $\alpha_1, \alpha_2, \cdots, \alpha_n$ has $\mathcal{M}^*(\underbrace{2, \cdots, 2, 2}_{n-1}) + 1$ mergings, and the number of mergings can't be reduced, implying

$$\mathcal{M}^*(\underbrace{2, 2, \cdots, 2}_{n}) \geq \mathcal{M}^*(\underbrace{2, \cdots, 2, 2}_{n-1}) + 1.$$

We thus prove the proposition.

■

Example III.11. It immediately follows from Proposition III.7 that

$$\mathcal{M}^*(1, 1, \cdots, 1) = 0.$$

Example III.12. It immediately follows from Proposition III.10 that

$$\mathcal{M}^*(\underbrace{2, 2, \cdots, 2}_{n}) = n - 1,$$

which was first shown in [4]. In particular, $\mathcal{M}(2, 2) = 1$. Further together with Proposition III.6, we have $\mathcal{M}^*(2, m) = 1$ for $m \geq 2$. Note that

$$\mathcal{M}^*(\underbrace{2, 2, \cdots, 2}_{n}) < \sum_{1 \leq i < j \leq n} \mathcal{M}^*(2, 2),$$

which implies the inequality in Theorem III.1 may not hold for certain cases.

Example III.13. It follows from Proposition III.9 that

$$\mathcal{M}^*(3, 3) \leq \mathcal{M}(2, 2) = 5.$$

One checks that the graph depicted by Figure 5 does not allow any rerouting to reduce the number of mergings, which implies $\mathcal{M}^*(3, 3) = 5$. Applying Theorem III.1, we have

$$\mathcal{M}^*(\underbrace{3, 3, \cdots, 3}_{n}) \leq \frac{5n(n-1)}{2}.$$

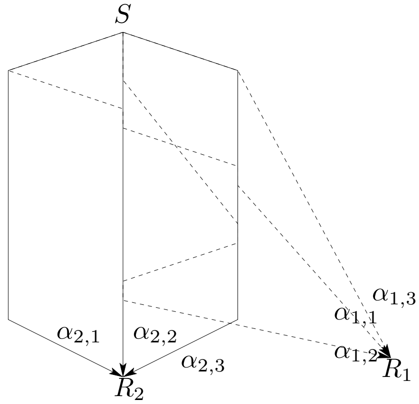

Fig. 5. an example achieving $\mathcal{M}^*(3, 3)$

IV. MOTIVATIONS

Mergings in directed graphs naturally relate to "congestions" of traffic flows in various networks. Particularly, in network coding theory [9], which studies digital communication networks carrying information flow [1], computations and estimations of $\mathcal{M}$ and $\mathcal{M}^*$ have drawn much interest recently. Recent related work in network coding theory listed in this section are done in very different languages; we shall briefly introduce network coding theory and describe these work using the terminology and notations in this paper.

Network coding is a novel technique to improve the capability of networks (directed graphs) to transfer digital information between senders (sources) and receivers (sinks). Before network coding, information is transferred among networks using the traditional routing scheme, where intermediate nodes (vertices) can only forward and duplicate the received information. In contrast to the routing scheme, the idea of network coding is to allow intermediate nodes to "combine" data received from different incoming links (edges), eventually boosting the transmission rate of the network.

For a very comprehensive introduction to network coding theory, we refer to [9]. Here, we roughly illustrate the idea of network coding using the following famous "butterfly network" [6]. Consider the network depicted in Figure 6, where each link has capacity 1 bit per time unit and there is no processing delay at each node. Two binary bits a, b are to be transmitted from the source S to Y and Z. If we ignore the transmission to Z, we can use path $S \to T \to Y$ to transmit a, and use path $S \to U \to W \to X \to Y$ to transmit b simultaneously; similarly ignoring the transmission to Y, we can use path $S \to U \to Z$ to transmit a, and use path $S \to T \to W \to X \to Z$ to transmit b simultaneously. Note that paths $S \to U \to W \to X \to Y$ and $S \to T \to W \to X \to Z$ merge at $W \to X$. If the traditional routing scheme is assumed, $W \to X$ will become

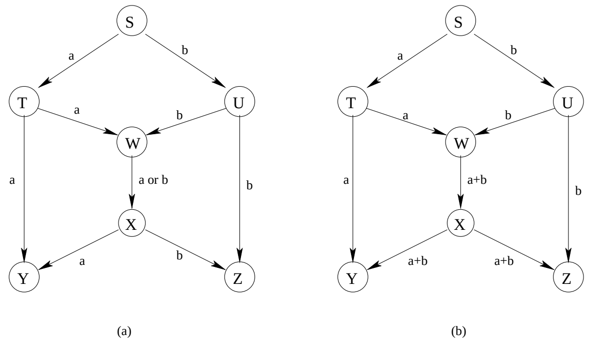

Fig. 6. network coding on the Butterfly network

a "bottleneck" for simultaneous data transmission to Y and Z, since for each time unit $W \rightarrow X$ can either carry a or b, but not both at once. Thus under the routing scheme, completion of data transmission takes at least 2 time units. Allowing intermediate nodes to recode the data from the incoming links, network coding scheme will provide a solution to speed up the data transmission: the "bottleneck" $W \rightarrow X$ carry a and b at the same time by carrying $a+b$, here $+$ denotes the exclusive-OR on a, b. Then as shown in Figure 6(b), Y will receive a and $a+b$, from which b can be decoded; at the same time unit Z will receive b and $a+b$, from which a can be decoded. In other words, with the encoding at node W, Y and Z can receive the complete data simultaneously within 1 time unit.

Now consider a general network with one sender S and n receivers $R_1, R_2, \cdots, R_n$, where each edge has capacity 1 bit per time unit and there is no processing delay at each node. Suppose that each R_i has the same min-cut c with the sender S, and c bit information are to be transmitted from S to all R_i's. Ignoring the presence of other receivers, any set of Menger's paths from S to a receiver is able to carry data to the receiver at the maximum possible rate c; however for simultaneous data transmission, any merging among these Menger's paths will become a bottleneck. It has been shown [1], [6] that with appropriate network coding at the merging nodes, all the receivers can receive the information at the maximum possible rate c.

In a network coding scheme, we call a node an "encoding node" if this node recodes the data from the incoming links, rather than simply duplicating and forwarding the incoming date. It is important to minimize, for a given network, the number of nodes which are needed to be equipped with such encoding capabilities, since these nodes are typically more expensive than other forwarding nodes, and may increase the overall complexity of the network. Since for given sets of Menger's paths from the source to the receivers, encoding operations are only needed at merging nodes among these paths, $\mathcal{M}$ and $\mathcal{M}^*$ with appropriate parameters will naturally give upper bounds on the number of necessary encoding nodes for a given network. In particular, for an acyclic network G with one source and multiple sinks, as suggested by Lemma 13 of [5], the minimum number of coding operations (required to guarantee all receivers receive data at the maximum possible rate) is equal to $M^*(G)$.

It was first conjectured that $\mathcal{M}(c_1, c_2, \cdots, c_n)$ is finite in [8]. More specifically the authors proved that (see Lemma 10 of [8]) if $\mathcal{M}(c_1, c_2)$ is finite for all c_1, c_2, then $\mathcal{M}(c_1, c_2, \cdots, c_n)$ is finite as well. To support the conjecture, the authors showed that $\mathcal{M}(2, c)$ is finite for any c, and subsequently $\mathcal{M}(\underbrace{2, 2, \cdots, 2}_{n}, c)$ is finite for any n and c. Lemma II.7 shows that indeed $\mathcal{M}(c_1, c_2)$ is finite for all c_1, c_2, thus the conjecture is true.

As for $\mathcal{M}^*$, the authors of [4] use the idea of "subtree decomposition" to first prove that

$$\mathcal{M}^*(\underbrace{2, 2, \cdots, 2}_{n}) = n - 1.$$

Although their idea seems to be difficult to generalize to other parameters, it does allow us to gain deeper understanding about the topological structure of minimum mergings achieving graph for this special case. It was first shown in [5] that $\mathcal{M}^*(c_1, c_2)$ is finite for all c_1, c_2 (see Theorem 22 of [5]), and subsequently $\mathcal{M}^*(c_1, c_2, \cdots, c_n)$ is finite all $c_1, c_2, \cdots, c_n$. The proof of Lemma II.7 is inspired by and follows closely the spirit of the proof of Theorem 22 of [5]. One of the differences between the approach in [5] and ours is that we start with arbitrarily chosen Menger's paths, and focus on transformations (more specifically, merging number reducing reroutings) of these paths, which allow us to see how $\mathcal{M}, \mathcal{M}^*$ depend on the min-cuts.

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