The Capacity of Markov Channels with Noiseless Output and State Feedback

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***Abstract*—In this paper, we derive a single-letter expression for the capacity of the finite-state channel (FSC) with delayed output and state feedback by formulating the problem in a stochastic control framework. The resulting capacity expression can be evaluated using dynamic programming. Although the considered channel is a special case of the one investigated by Tatikonda and Mitter, our approach is somewhat different and the resulting capacity expression is significantly simpler. This single-letter characterization of the capacity is the first step in investigating posterior matching-like transmission schemes for the case of FSCs.**

I. INTRODUCTION

Communication in the presence of feedback has been a long studied problem which dates back to Shannon's early work [1], where he proved that feedback cannot increase the capacity of memoryless channels. Feedback, however, can improve the error performance and/or can simplify the transmission scheme. Horstein [2] proposed a simple sequential transmission scheme which is capacity-achieving and provides larger error exponents than traditional fixed-length block-coding. Similarly, Schalkwijk and Kailath [3] showed that capacity and a double exponentially decreasing error probability can be achieved by a simple sequential transmission scheme for the additive white Gaussian noise channel (AWGNC) with average power constraint. Recently, Shayevitz and Feder [4], [5] identified an underlying principle shared by the aforementioned Horstein and Schalkwijk-Kailath schemes and introduced a simple encoding scheme, namely the posterior matching scheme (PMS) for general memoryless channels. Furthermore, they showed that the PMS achieves the capacity of general DMCs. Subsequently, Coleman [6] revisited the PMS and provided a proof of capacity achievability by reformulating the problem in a stochastic control framework.

In our work we would like to generalize the PMS for finite state channels (FSC) where the channel state is affected both by nature and by the input sequence (thus introducing inter-symbol interference (ISI)), the channel state information (CSI) and output are available at the receiver, and the CSI and output are available at the transmitter with unit delay through noiseless feedback. The starting point of our investigation and the contribution of this paper is the derivation of a single-letter capacity expression for this channel.

One of the first capacity results for FSCs was by Viswanathan [7], who found the capacity of a FSC with receiver CSI and delayed feedback where there is no ISI. Later, Chen and Berger [8] found the capacity of a FSC with ISI where current CSI is available at the transmitter and the receiver. Yang et. al. [9] used a stochastic control method to find the capacity of the ISI channel. Recently, Tatikonda and Mitter [10] provided a general stochastic control framework for evaluating the capacity of the FSC with feedback. In that paper, the capacity was characterized as the solution of a dynamic programming average cost optimality equation (ACOE). Como et. al. [11] used an approach similar to [10] to find the capacity of the FSC when current CSI is available at the transmitter and the receiver. An upper bound on the capacity of the FSC without ISI and CSI was found using dynamic programming by Huang et. al. [12]. We point out that although the channel considered in this paper is indeed a special case of the one considered in [10], our approach in deriving a single-letter capacity expression is somewhat different and the resulting capacity expression is significantly simpler.

The remainder of the paper is organized as follows. In Section II, the channel model and the general form of the capacity are introduced. We derive a simplified single-letter expression of the capacity in Section III. Section IV concludes the paper.

II. CHANNEL MODEL AND PRELIMINARIES

We consider channels with input X_t, output Y_t and state S_t at time t. The corresponding input, output and state random processes are denoted by $(X_t)_{t=1}^{\infty}$, $(Y_t)_{t=1}^{\infty}$, $(S_t)_{t=1}^{\infty}$, respectively. Input, output and state alphabets are finite and of size $|\mathcal{X}| = K_x$, $|\mathcal{Y}| = K_y$, $|\mathcal{S}| = K_s$, respectively. At time t the receiver has access to the current channel output y_t and state s_t. The state s_t and output y_t are fed back to the transmitter with unit delay. Let $P(x^T, s^T, y^T)$ be the joint probability mass function (pmf) of X^T, Y^T, and s^T, where v^T denotes the length-T vector $(v_1, \ldots, v_T)$. Then,

$$P(x^T, s^T, y^T) = Q'(y_1|x_1, s_1)Q(s_1)P(x_1) \prod_{t=2}^{T} Q'(y_t|x_t, s_t)Q(s_t|s_{t-1}, x_{t-1})P(x_t|x^{t-1}, s^{t-1}, y^{t-1}). \quad (1)$$

Implicit in (1), is the fact that the considered channel states are affected by both nature and ISI.

A sequence of joint measures $\{P(x^T, s^T, y^T)\}_{T=1}^{\infty}$ is *directed information stable* if

$$\lim_{T\to\infty} P(|\frac{\overrightarrow{i}(X^T; S^T, Y^T)}{I(X^T \to S^T, Y^T)} - 1| > \epsilon) = 0, \qquad \forall \epsilon > 0, \quad (2)$$

where $\overrightarrow{i}(X^T; S^T, Y^T) = \log \frac{P(x^T|s^T,y^T)}{\prod_{t=1}^T P(x^t|s^{t-1},y^{t-1})}$ and $I(X^T \to S^T, Y^T) = \sum_{t=1}^T I(X^t; S_t, Y_t|S^{t-1}, Y^{t-1})$. Throughout the paper we assume directed information stability.

In [10] the authors have developed a capacity expression for the general class of such channels in the form of

$$C = \sup_{\{\{P(x_t|x^{t-1},s^{t-1},y^{t-1})\}_{t=1}^T\}_{T=1}^{\infty}} \liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^T I(X^t; S_t, Y_t|S^{t-1}, Y^{t-1}). \quad (3)$$

This expression was further simplified in [10] to

$$C = \sup_{\{\{P(x_t|\pi_t,\gamma_t,s_{t-1})\}_{t=1}^T\}_{T=1}^{\infty}} \liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^T I(X_t, \Pi_t; S_t, Y_t|S_{t-1}, \Gamma_t) \quad (4)$$

$$= \sup_{\{P(X|\Pi,\Gamma,S')\}_{\Pi,\Gamma,S'}} I(X, \Pi; S, Y|S', \Gamma) \quad (5)$$

where $\Pi_t \in \mathcal{P}(\mathcal{S})$ defined as $\Pi_t(s_t) \stackrel{\text{def}}{=} P(s_t|X^{t-1}, S^{t-1}, Y^{t-1})$, and $\Gamma_t \in \mathcal{P}(\mathcal{P}(\mathcal{S}))$ defined as $\Gamma_t(\pi_t) \stackrel{\text{def}}{=} P(\pi_t|S^{t-1}, Y^{t-1})$. In the above, the notation $\mathcal{P}(\mathcal{S})$ is used to denote the set of probability measures on the set S. It was further shown in [10] that the capacity expression can in principle be evaluated as the solution of an appropriate ACOE [13, Th. 6.2,Th. 6.3].

Since the above expressions are developed for general feedback patterns, and since we are only interested in the special case where both state and output are fed back to the transmitter with unit delay, it may be possible to further simplify the capacity expression in (4). We will pursue this direction in the following section.

III. A SIMPLIFIED SINGLE-LETTER CAPACITY EXPRESSION

Consider the term $I(X^t; S_t, Y_t|S^{t-1}, Y^{t-1})$ with the channel input distribution $P(x_t|x^{t-1}, s^{t-1}, y^{t-1})$ in (3). The following theorem proves that the form of the optimal channel input distribution can be simplified.

Lemma 1. *For every T,*

$$\sup_{\{P(x_t|x^{t-1},s^{t-1},y^{t-1})\}_{t=1}^T} \frac{1}{T}\sum_{t=1}^T I(X^t; S_t, Y_t|S^{t-1}, Y^{t-1}) = \sup_{\{P(x_t|x_{t-1},s^{t-1},y^{t-1})\}_{t=1}^T} \frac{1}{T}\sum_{t=1}^T I(X_{t-1}^t; S_t, Y_t|S^{t-1}, Y^{t-1}) \quad (6)$$

Proof: First, note that for every t

$$\begin{aligned} &I(X^t; S_t, Y_t|S^{t-1}, Y^{t-1}) = \\ &= I(X_{t-1}^t; S_t, Y_t|S^{t-1}, Y^{t-1}) + \\ &\quad + I(X^{t-2}; S_t, Y_t|S^{t-1}, Y^{t-1}, X_{t-1}^t) \quad (7a) \\ &\stackrel{(a)}{=} I(X_{t-1}^t; S_t, Y_t|S^{t-1}, Y^{t-1}). \quad (7b) \end{aligned}$$

where (a) is due to the fact that S_t, Y_t is independent of X^{t-2} conditioned on $S^{t-1}, Y^{t-1}, X_{t-1}^t$.

Each of the terms $I(X_{t-1}^t; S_t, Y_t|S^{t-1}, Y^{t-1})$ in the summation is evaluated based on the joint distribution $P(x_{t-1}^t, s^t, y^t)$. We now proceed by induction to prove that the sequence of measures $\{P(x^t, s^t, y^t)\}_{t=1}^T$ induced by the sequence of channel input distributions $\{P(x_t|x^{t-1}, s^{t-1}, y^{t-1})\}_{t=1}^T$ equals to the sequence of measures $\{P_1(x^t, s^t, y^t)\}_{t=1}^T$ induced by an appropriately defined sequence of channel input distributions of the form $\{P_1(x_t|x_{t-1}, s^{t-1}, y^{t-1})\}_{t=1}^T$.

For $t = 1$ we set $P_1(x_1) = P(x)$ and have

$$\begin{aligned} P_1(x_1, s_1, y_1) &= Q'(y_1|s_1, x_1)Q(s_1)P_1(x_1) \quad (8a) \\ &= Q'(y_1|s_1, x_1)Q(s_1)P(x_1) = P(x_1, s_1, y_1). \quad (8b) \end{aligned}$$

Now for $t + 1$ we set $P_1(x_{t+1}|x_t, s^t, y^t) = P(x_{t+1}|x_t, s^t, y^t) = \frac{\sum_{x^{t-1}} P(x^{t+1}|x^t,s^t,y^t)P(x^t,s^t,y^t)}{\sum_{x^{t-1}} P(x^t,s^t,y^t)}$ and have

$$\begin{aligned} &P_1(x^{t+1}, s^{t+1}, y^{t+1}) \\ &\quad = Q'(y_{t+1}|s_{t+1}, x_{t+1})Q(s_{t+1}|s_t, x_t) \\ &\qquad \times P_1(x_{t+1}|x_t, s^t, y^t)P_1(x^t, s^t, y^t) \quad (9a) \\ &\quad \stackrel{(a)}{=} Q'(y_{t+1}|s_{t+1}, x_{t+1})Q(s_{t+1}|s_t, x_t) \\ &\qquad \times P(x_{t+1}|x_t, s^t, y^t)P(x^t, s^t, y^t) \quad (9b) \\ &\quad = P(x^{t+1}, s^{t+1}, y^{t+1}), \quad (9c) \end{aligned}$$

where (a) is due to the construction of $P_1(x_{t+1}|x_t, s^t, y^t)$, and the induction hypothesis. The above equality implies that the marginal measures are also equal, i.e., $P_1(x_t^{t+1}, s^{t+1}, y^{t+1}) = P(x_t^{t+1}, s^{t+1}, y^{t+1})$ for all $t = 1, \ldots, T$, which further implies the equality in (6). ■

The above lemma shows that in order to achieve capacity it is sufficient to restrict the channel input distributions to be the form of $P(x_t|x_{t-1}, s^{t-1}, y^{t-1})$, i.e., the capacity expression becomes

$$C = \sup_{\{\{P(x_t|x_{t-1},s^{t-1},y^{t-1})\}_{t=1}^T\}_{T=1}^{\infty}} \liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^T I(X^t; S_t, Y_t|S^{t-1}, Y^{t-1}). \quad (10)$$

To further simplify the capacity expression, we will formulate a control problem which is equivalent to the problem of computing capacity. Towards this end, let $(x_{t-1}, s_{t-1}, y_{t-1})$ be the system state at time t, and (s_{t-1}, y_{t-1}) be the controller observation at time t. Let the control action at time t be U_t :

$\mathcal{X} \to \mathcal{P}(\mathcal{X})$ defined as $u_t(x_t|x_{t-1}) \stackrel{\text{def}}{=} P(x_t|x_{t-1}, s^{t-1}, y^{t-1})$, i.e., the control action at time t can be a function of all observations s^{t-1}, y^{t-1} up to time t. Further, define the instantaneous reward at time t to be $R_t = \log \frac{P(S_t, Y_t|S^{t-1}, Y^{t-1}, X_{t-1}^t)}{P(S_t, Y_t|S^{t-1}, Y^{t-1})}$. The control problem is to determine the optimal policy $g = \{g_t\}_{t=1}^\infty$ (such that $u_t = g_t(s^{t-1}, y^{t-1})$) that maximizes the average expected reward $\liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^T E^g[R_t]$.

First we need to prove that the above control problem is equivalent to the problem of computing capacity as stated in (10). In view of the fact that $E^g[R_t] = I(X^t; S_t, Y_t|S^{t-1}, Y^{t-1})$, this equivalence is established in the following lemma.

Lemma 2. *For every sequence of channel input distributions $\{P(x_t|x_{t-1}, s^t, y^t)\}_{t=1}^\infty$ with resulting sequence of joint measures $\{P(x_{t-1}^t, s^t, y^t)\}_{t=1}^\infty$ there exists a policy g with resulting sequence of joint measures $\{P^g(x_{t-1}^t, s^t, y^t)\}_{t=1}^\infty$ such that for each t: $P^g(x_{t-1}^t, s^t, y^t) = P(x_{t-1}^t, s^t, y^t)$. Conversely, for every policy g with resulting sequence of joint measures $\{P^g(x_{t-1}^t, s^t, y^t)\}_{t=1}^\infty$ there exists a sequence of channel input distributions $\{P(x_t|x_{t-1}, s^t, y^t)\}_{t=1}^\infty$ with resulting sequence of joint measures $\{P(x_{t-1}^t, s^t, y^t)\}_{t=1}^\infty$ such that for each t: $P(x_{t-1}^t, s^t, y^t) = P^g(x_{t-1}^t, s^t, y^t)$.*

Proof: We will use the notation $u_t = g_t(s^{t-1}, y^{t-1})$ and for convenience, we will write $u_t(x_t|x_{t-1}) = g_t[s^{t-1}, y^{t-1}](x_t|x_{t-1})$.

For the direct part, for each t we choose a policy g_t as

$$g_t[s^{t-1}, y^{t-1}](x_t|x_{t-1}) = P(x_t|x_{t-1}, s^{t-1}, y^{t-1}), \qquad (11)$$

and proceed by induction.

For $t = 1$ we have

$$\begin{aligned} P^g(x_1, s_1, y_1) &= Q'(y_1|s_1, x_1)Q(s_1)g_1(x_1) & (12a)\\ &= Q'(y_1|s_1, x_1)Q(s_1)P(x_1) = P(x_1, s_1, y_1). & (12b)\end{aligned}$$

Now for $t + 1$ we have

$$\begin{aligned} &P^g(x_t^{t+1}, s^{t+1}, y^{t+1})\\ &\quad= Q'(y_{t+1}|x_{t+1}, s_{t+1})Q(s_{t+1}|s_t, x_t)\\ &\qquad\times g_{t+1}[s^t, y^t](x_{t+1}|x_t)\sum_{x_{t-1}} P^g(x_{t-1}^t, s^t, y^t) & (13a)\\ &\quad\stackrel{(a)}{=} Q'(y_{t+1}|x_{t+1}, s_{t+1})Q(s_{t+1}|s_t, x_t)\\ &\qquad\times P(x_{t+1}|x_t, s^t, y^t)\sum_{x_{t-1}} P(x_{t-1}^t, s^t, y^t) & (13b)\\ &\quad= P(x_t^{t+1}, s^{t+1}, y^{t+1}), & (13c)\end{aligned}$$

where (a) is due to the choice of the policy g_{t+1} and the induction hypothesis.

For the converse, for each t we choose a channel input distribution as

$$\begin{aligned} P(x_t|x_{t-1}, s^{t-1}, y^{t-1}) &= P^g(x_t|x_{t-1}, s^{t-1}, y^{t-1}) & (14a)\\ &= g_t[s^{t-1}, y^{t-1}](x_t|x_{t-1}). & (14b)\end{aligned}$$

Then, for $t = 1$ we have

$$\begin{aligned} P(x_1, s_1, y_1) &= Q'(y_1|s_1, x_1)Q(s_1)P(x_1) & (15a)\\ &= Q'(y_1|s_1, x_1)Q(s_1)g_1(x_1) = P^g(x_1, s_1, y_1). & (15b)\end{aligned}$$

Now for $t + 1$ we have

$$\begin{aligned} &P(x_t^{t+1}, s^{t+1}, y^{t+1})\\ &\quad= Q'(y_{t+1}|x_{t+1}, s_{t+1})Q(s_{t+1}|s_t, x_t)\\ &\qquad\times P(x_{t+1}|x_t, s^t, y^t)\sum_{x_{t-1}} P(x_{t-1}^t, s^t, y^t) & (16a)\\ &\quad\stackrel{(a)}{=} Q'(y_{t+1}|x_{t+1}, s_{t+1})Q(s_{t+1}|s_t, x_t)\\ &\qquad\times P^g(x_{t+1}|x_t, s^t, y^t)\sum_{x_{t-1}} P^g(x_{t-1}^t, s^t, y^t) & (16b)\\ &\quad= P^g(x_t^{t+1}, s^{t+1}, y^{t+1}) & (16c)\end{aligned}$$

where (a) is due to the construction of the channel input distributions and the induction hypothesis. ■

We are now ready to state and prove the main result of this section.

Proposition 1. *The capacity of the finite-state channel with feedback defined in Section II is*

$$C = \sup_{\{P(X|X', S', \Theta)\}_{X', S', \Theta}} I(X, X'; S, Y|S', \Theta) \qquad (17)$$

where $\Theta \in \mathcal{P}(\mathcal{X})$, and the mutual information is evaluated using the joint measure

$$\begin{aligned} &P(Y, S, X, X', S', d\Theta)\\ &= Q'(Y|S, X)Q(S|S', X')P(X|X', S', \Theta)\Theta(X')P(S', d\Theta). & (18a)\end{aligned}$$

The distribution $P(S, d\Theta)$ is the solution of the equation

$$\begin{aligned} &P(S, d\Theta')\\ &= \int_{S', \Theta} P(S', d\Theta)\sum_Y \delta_{\phi(\Theta, P(X|X', S', \Theta), Y, S, S')}(\Theta')\times\\ &\qquad \sum_X Q'(Y|X, S)\sum_{X'} Q(S|S', X')P(X|X', S', \Theta)\Theta(X), & (18b)\end{aligned}$$

where the function $\phi(\cdot)$ is defined in the following proof.

Proof: We consider the stochastic control problem described above. The system state $(x_{t-1}, s_{t-1}, y_{t-1})$ evolves as a controlled Markov chain with control action u_t because

$$\begin{aligned} &P(x_t, s_t, y_t|x^{t-1}, s^{t-1}, y^{t-1}, u^t)\\ &\quad= Q'(y_t|s_t, x_t)Q(s_t|x_{t-1}, s_{t-1})u_t(x_t|x_{t-1}) & (19a)\\ &\quad= P(x_t, s_t, y_t|x_{t-1}, s_{t-1}, y_{t-1}, u_t) & (19b)\end{aligned}$$

Define the information state $\Theta_t \in \mathcal{P}(\mathcal{X})$ with $\Theta_t(x_t) \stackrel{\text{def}}{=} P(x_t|S^t, Y^t)$ as the posterior belief on the unobserved part of the system state. Then,

$$\begin{aligned}
&\theta_t(x_t)\\
&= P(x_t|s^t, y^t) && \text{(20a)}\\
&= \frac{P(x_t, s_t, y_t|s^{t-1}, y^{t-1})}{P(s_t, y_t|s^{t-1}, y^{t-1})} && \text{(20b)}\\
&= \Big(\sum_{x_{t-1}} Q'(y_t|x_t, s_t) Q(s_t|x_{t-1}, s_{t-1}) u_t(x_t|x_{t-1})\\
&\quad \times P(x_{t-1}|s^{t-1}, y^{t-1}) \Big) / \sum_{x_t} P(x_t, s_t, y_t|s^{t-1}, y^{t-1}) && \text{(20c)}\\
&= \Big(\sum_{x_{t-1}} Q'(y_t|x_t, s_t) Q(s_t|x_{t-1}, s_{t-1}) u_t(x_t|x_{t-1})\\
&\quad \times \theta_{t-1}(x_{t-1}) \Big) / \Big(\sum_{x_t} \sum_{x_{t-1}} Q'(y_t|x_t, s_t)\\
&\quad \times Q(s_t|x_{t-1}, s_{t-1}) u_t(x_t|x_{t-1}) \theta_{t-1}(x_{t-1}) \Big), && \text{(20d)}
\end{aligned}$$

which implies that θ_t can be recursively updated as $\theta_t = \phi(\theta_{t-1}, u_t, y_t, s_t, s_{t-1})$.

We now show that $\{(S_{t-1}, \Theta_{t-1})\}_t$ is a controlled Markov chain with control U_t. Indeed,

$$\begin{aligned}
&P(s_t, d\theta_t|s^{t-1}, \theta^{t-1}, u^t)\\
&= \sum_{y_t} P(d\theta_t|y_t, s^t, \theta^{t-1}, u^t) P(s_t, y_t|s^{t-1}, \theta^{t-1}, u^t) && \text{(21a)}\\
&= \sum_{y_t} \delta_{\phi(\theta_{t-1}, u_t, y_t, s_t, s_{t-1})}(\theta_t) \sum_{x_t} Q'(y_t|x_t, s_t)\\
&\quad \times \sum_{x_{t-1}} Q(s_t|s_{t-1}, x_{t-1}) u_t(x_t|x_{t-1}) \theta_{t-1}(x_{t-1}) && \text{(21b)}\\
&= P(s_t, d\theta_t|s_{t-1}, \theta_{t-1}, u_t). && \text{(21c)}
\end{aligned}$$

Furthermore, the instantaneous reward r_t can be written as

$$\begin{aligned}
&r_t\\
&= \log \frac{P(s_t, y_t|s^{t-1}, y^{t-1}, x_{t-1}^t)}{P(s_t, y_t|s^{t-1}, y^{t-1})} && \text{(22a)}\\
&= \log \Big\{ \Big(Q'(y_t|x_t, s_t) Q(s_t|s_{t-1}, x_{t-1}) \Big) / \Big(\sum_{x_{t-1}^t} Q'(y_t|x_t, s_t)\\
&\quad \times Q(s_t|s_{t-1}, x_{t-1}) P(x_t|x_{t-1}, s^{t-1}, y^{t-1})\\
&\quad \times P(x_{t-1}|s^{t-1}, y^{t-1}) \Big) \Big\} && \text{(22b)}\\
&= \log \Big\{ \Big(Q'(y_t|x_t, s_t) Q(s_t|s_{t-1}, x_{t-1}) \Big) / \Big(\sum_{x_{t-1}^t} Q'(y_t|x_t, s_t)\\
&\quad \times Q(s_t|s_{t-1}, x_{t-1}) u_t(x_t|x_{t-1}) \theta_{t-1}(x_{t-1}) \Big) \Big\}. && \text{(22c)}
\end{aligned}$$

The expected reward at time t conditioned on the states and control actions up to time t is

$$\begin{aligned}
&E\left[R_t|s^{t-1}, \theta^{t-1}, u^t\right]\\
&= E\Big[\log \Big\{ \Big(Q'(Y_t|X_t, S_t) Q(S_t|s_{t-1}, X_{t-1}) \Big)\\
&\qquad / \Big(\sum_{x_{t-1}^t} Q'(Y_t|x_t', S_t) Q(S_t|s_{t-1}, x_{t-1}')\\
&\qquad \times u_t(x_t'|x_{t-1}') \theta_{t-1}(x_{t-1}') \Big) \Big\} \mid s^{t-1}, \theta^{t-1}, u^t \Big] && \text{(23a)}\\
&= \sum_{y_t, x_{t-1}^t, s_t} Q'(y_t|x_t, s_t) Q(s_t|s_{t-1}, x_{t-1})\\
&\qquad \times u_t(x_t|x_{t-1}) \theta_{t-1}(x_{t-1})\\
&\qquad \times \log \Big\{ \Big(Q'(y_t|x_t, s_t) Q(s_t|s_{t-1}, x_{t-1}) \Big)\\
&\qquad / \Big(\sum_{x_{t-1}^t} Q'(y_t|x_t', s_t) Q(s_t|s_{t-1}, x_{t-1}')\\
&\qquad \times u_t(x_t'|x_{t-1}') \theta_{t-1}(x_{t-1}') \Big) \Big\} && \text{(23b)}\\
&= \overline{r}(s_{t-1}, \theta_{t-1}, u_t), && \text{(23c)}
\end{aligned}$$

which is only a function of the observed part of the state s_{t-1}, the belief on the unobserved part of the state θ_{t-1}, and the action u_t.

We can now deduce from standard results in Markov decision processes that the optimal policy at time t need only be a function of (s_{t-1}, θ_{t-1}), i.e., the optimizing distributions in (10) need only be of the form $P(x_t|x_{t-1}, s_{t-1}, \theta_{t-1})$ and the capacity expression becomes

$$C = \sup_{\{P(x_t|x_{t-1}, \theta_{t-1}, s_{t-1})\}_t} \liminf_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} I(X_{t-1}^t; S_t, Y_t|S_{t-1}, \Theta_{t-1}). \tag{24}$$

Note that above described controlled Markov chain is time-homogenous, and hence the optimal channel input distribution is time-invariant, and consequently the capacity expression reduces to the one in (17). ■

To find the capacity using (17), we need to identify the stationary distribution of S and Θ for each choice of $P(X|X', S', \Theta)$ using (18b), and evaluate the mutual information by using the joint measure specified in (18a). Alternatively, we may use dynamic programming to find the capacity. In other words, the optimal value in (17) can be obtained by the solution of the following ACOE with some bounded function $\eta : \mathcal{S} \times \Theta \to \mathcal{R}$ [13, Th. 6.2, Th. 6.3]

$$C + \eta(s, \theta) = \sup_u J(\tilde{s}, s, \theta, u), \tag{25}$$

where

$$J(\tilde{s}, s, \theta, u) = \overline{r}(s, \theta, u) + \Bigg(\sum_{\tilde{s}, \tilde{y}} \eta(\tilde{s}, \phi(\theta, u, \tilde{y}, \tilde{s}, s)) \sum_{\tilde{x}} Q'(\tilde{y}|\tilde{s}, \tilde{x}) \times \sum_{x} Q(\tilde{s}|s, x) u(\tilde{x}|x) \theta(x) \Bigg). \tag{26}$$

IV. Conclusion

A single-letter expression for the capacity of the FSC with delayed feedback was derived. The methodology was based on reformulating the capacity problem as a stochastic control problem. This methodology is quite general and will likely be useful in finding single-letter expressions for the capacity of other channels, such as the multiple-access channel with feedback.

In a subsequent paper [14], a simple recursive transmission scheme is presented that can be thought of as the generalization of the PMS.

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