On Interactive Encoding and Decoding for Distributed Lossless Coding of Individual Sequences

En-Hui Yang and Jin Meng
Department of Electrical and
Computer Engineering
University of Waterloo
Waterloo, Ontario N2L 6P6
Email: {ehyang,j4meng}@uwaterloo.ca

***Abstract*—Distributed near lossless coding of individual sequences X and Y is considered, where X and Y are first encoded separately and then sent to a joint decoder. Unlike distributed near lossless coding of correlated random sources, the joint decoder in distributed coding of individual sequences does not help at all. In other words, the minimum numbers of bits to be sent from X and Y respectively to the joint decoder are the same as in two independent, parallel systems where X and Y are encoded separately and decoded separately. In this paper, however, we show that by using interactive encoding and decoding where the joint decoder is allowed to interact with both separate encoders, the minimum number of total bits to be exchanged between the joint decoder and two separate encoders for each and every pair of individual sequences X and Y is the same as in the system where X and Y are jointly encoded and then jointly decoded, while X and Y can be recovered by the joint decoder in a near lossless manner.**

I. Introduction

Consider distributed near lossless coding of two sources X and Y, where X and Y are first encoded separately and then sent to a joint decoder. When X and Y are random, correlated, and memoryless sources, Slepian and Wolf showed in their seminar work [1] that the minimum number of total bits per symbol pair needed to be unidirectionally transmitted from two separated encoders to the joint decoder in order for (X, Y) to be recovered with high probability by the joint decoder is still equal to the joint entropy, the minimum number of transmitted bits in the system where X and Y are jointly encoded and then jointly decoded. The same result was later shown in [2], [3] to be valid as well for stationary ergodic sources (X, Y). In this paper, we shall refer to these results collectively as the Slepian-Wolf result and to the corresponding type of distributed coding paradigm as Slepian-Wolf coding (SWC).

The Slepian-Wolf result, however, does not hold any more when X and Y are individual sequences. In other words, in Slepian-Wolf coding of individual sequences, the joint decoder does not help at all, and the minimum numbers of bits to be sent unidirectionally from X and Y respectively to the joint decoder are the same as in two independent, parallel systems where X and Y are encoded separately and decoded separately.

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada under Grants RGPIN203035-02 and RGPIN203035-06, and by the Canada Research Chairs Program.

To overcome the above problem, in this paper, we consider interactive encoding and decoding (IED) for distributed near lossless coding of individual sequences as shown in Figure 1, where two individual sequences $X = x^n$ and $Y = y^n$ are encoded separately, and decoded jointly, and where the joint decoder is allowed to interact with two separate encoders. The concept of IED was first formalized in [4], [5] with emphasis on IED for near lossless source coding with decoder side information, where one source is transmitted from the encoder to the decoder, and another source is available only to the decoder as side information. By measuring the performance of an IED scheme in terms of the average number of bits per symbol exchanged between the encoder and the decoder, the decoding error probability, and the number of interactions, it was shown in [4], [5] that IED is superior to Slepian-Wolf coding in several aspects for near lossless source coding with decoder side information. In particular, universal IED schemes for near lossless source coding with decoder side information can be constructed which can achieve the conditional entropy rate for any stationary source pair without knowing the statistics of the sources. On the other hand, no such SWC schemes exist.

In this paper, we extend the advantage of IED over SWC to the source coding network shown in Figure 1 and show that by using IED, the minimum number of total bits to be exchanged between the joint decoder and two separate encoders for each and every pair of individual sequences X and Y is the same as in the system where X and Y are jointly encoded and then jointly decoded, while X and Y can be recovered by the joint decoder in a near lossless manner. Specifically, given any four classical source coding schemes—one scheme encoding x^n alone, one scheme encoding y^n conditionally given x^n, one scheme encoding y^n alone, and one scheme encoding x^n conditionally given y^n—we demonstrate how to construct a universal random IED scheme for Figure 1 such that for each and every pair of individual sequences x^n and y^n, the number of total bits to be exchanged between the joint decoder and two separate encoders of the IED scheme is roughly the same as in the joint encoding and decoding of x^n and y^n (via the encoding of x^n plus the conditional encoding of y^n given x^n or the encoding of y^n plus the conditional encoding of x^n given y^n), and at the same time, x^n and y^n

can be recovered by the joint decoder of the IED scheme with high probability. This is in sharp contrast to the case of SWC where for individual sequences, there is no better SWC scheme than two independent parallel systems, where x^n and y^n are separately encoded and decoded. Therefore, our result implies that allowing interactions strictly improves the coding performance for distributed lossless coding of individual sequences. In addition, it is shown that by changing the parameters of the universal random IED scheme, we can adjust the number of bits of each link while keeping the number of total bits exchanged between the joint decoder and two encoders intact.

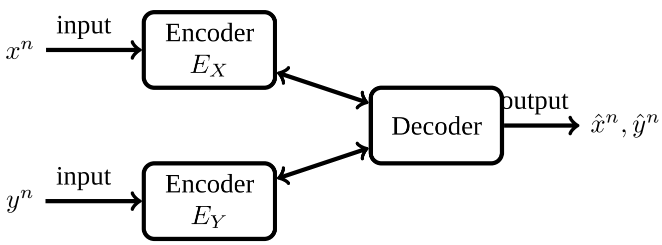

Fig. 1. Interactive encoding and decoding for distributed lossless coding

The rest of the paper is organized as follows. Section II introduces definitions and convention to be used later in our discussion. Section III presents the universal random IED scheme along with its performance analysis. Finally, conclusions will be drawn in Section IV.

II. Definitions and Convention

To facilitate our subsequent discussion, in this section, we introduce several definitions and convention.

For any sequence x^n over alphabet $\mathcal{X}$, x_B^n is the binary sequence of length $n\lceil\log|\mathcal{X}|\rceil$ generated by converting each symbol of x^n into its binary representation of length $\lceil\log|\mathcal{X}|\rceil$. The notation $|S|$ stands for the length of S if S is a sequence, and the cardinality of S if S is a finite set.

Let $\mathcal{C}_n = \{f_n, g_n\}$ be a classical lossless coding scheme of order n over alphabet $\mathcal{Z}$, where f_n is an encoder encoding sequences of length n over alphabet $\mathcal{Z}$ into binary sequences from a binary prefix set, and g_n is its corresponding decoder. Then the normalized codeword length function l_n of $\mathcal{C}_n$ is defined as

$$l_n(z^n) = \frac{|f_n(z^n)|}{n} .$$

Let $A^{l/n}$ be the set of sequences z^n, defined as

$$A^{l/n} \triangleq \{z^n : l_n(z^n) \le l/n\} .$$

The following lemma is proved in [5].

Lemma 1: For any lossless coding scheme $\mathcal{C}_n$ of order n, and any $l \in (0, n\lceil\log|\mathcal{Z}|\rceil)$,

$$|A^{l/n}| \le 2^l .$$

Pick any four classical lossless coding schemes $\mathcal{C}_n^{(X)}$, $\mathcal{C}_n^{(Y)}$, $\mathcal{C}_n^{(X|Y)}$, and $\mathcal{C}_n^{(Y|X)}$, where $\mathcal{C}_n^{(X)}$ and $\mathcal{C}_n^{(Y)}$ are classical lossless coding schemes of order n over alphabets $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $\mathcal{C}_n^{(X|Y)}$ ($\mathcal{C}_n^{(Y|X)}$, respectively) is classical conditional lossless coding scheme with its source over alphabet $\mathcal{X}$ ($\mathcal{Y}$, respectively) and side information over alphabet $\mathcal{Y}$ ($\mathcal{X}$, respectively) available to both the encoder and decoder. Note that given each y^n, $\mathcal{C}_n^{(X|Y)}$ is a lossless code of order n over alphabet $\mathcal{X}$ with y^n serving as side information available to both its encoder and decoder. Likewise, given each x^n, $\mathcal{C}_n^{(Y|X)}$ is a lossless code of order n over alphabet $\mathcal{Y}$ with x^n serving as side information available to both its encoder and decoder. Accordingly, with respect to $\mathcal{C}_n^{(X)}$, $\mathcal{C}_n^{(Y)}$, $\mathcal{C}_n^{(X|Y)}$, and $\mathcal{C}_n^{(Y|X)}$, the respective normalized (unconditional and conditional) codeword length functions $l_n^{(X)}(\cdot)$, $l_n^{(Y)}(\cdot)$, $l_n^{(X|Y)}(\cdot|y^n)$ and $l_n^{(Y|X)}(\cdot|x^n)$, as well as sets $A_X^{l/n}$, $A_Y^{l/n}$, $A_{X|Y}^{l/n}(y^n)$ and $A_{Y|X}^{l/n}(x^n)$, are well defined. It is easy to check that Lemma 1 applies to any of $A_X^{l/n}$, $A_Y^{l/n}$, $A_{X|Y}^{l/n}(y^n)$, and $A_{Y|X}^{l/n}(x^n)$. Further define

$$A_{XY}^{l/n} \triangleq \{(x^n, y^n) : l_n^{(X)}(x^n) + l_n^{(Y|X)}(y^n|x^n) \le l/n\}$$
$$A_{YX}^{l/n} \triangleq \{(x^n, y^n) : l_n^{(Y)}(y^n) + l_n^{(X|Y)}(x^n|y^n) \le l/n\}$$

Since $l_n^{(X)}(x^n) + l_n^{(Y|X)}(y^n|x^n)$ is the normalized codeword length function of the joint encoding of x^n and y^n via the encoding of x^n by $\mathcal{C}_n^{(X)}$ plus the conditional encoding of y^n given x^n by $\mathcal{C}_n^{(Y|X)}$, Lemma 1 applies to $A_{XY}^{l/n}$. Similarly, Lemma 1 applies to $A_{XY}^{l/n}$ as well. For convenience, we also define

$$L_n^{(M)}(x^n, y^n) \triangleq \max\left\{ \begin{array}{l} l_n^{(X)}(x^n) + l_n^{(Y|X)}(y^n|x^n) \\ l_n^{(Y)}(y^n) + l_n^{(X|Y)}(x^n|y^n) \end{array} \right\}$$
$$L_n^{(m)}(x^n, y^n) \triangleq \min\left\{ \begin{array}{l} l_n^{(X)}(x^n) + l_n^{(Y|X)}(y^n|x^n) \\ l_n^{(Y)}(y^n) + l_n^{(X|Y)}(x^n|y^n) \end{array} \right\}$$

With reference to Figure 1, let $\mathcal{I}_n$ be a deterministic IED scheme of order n. Given any x^n and y^n, we use $R_f^{(X)}(x^n, y^n|\mathcal{I}_n)$ and $R_f^{(Y)}(x^n, y^n|\mathcal{I}_n)$ to represent the number of bits per symbol in the forward direction of each link (from the encoders E_X and E_Y to the decoder, respectively), and $R_b^{(X)}(x^n, y^n|\mathcal{I}_n)$ and $R_b^{(Y)}(x^n, y^n|\mathcal{I}_n)$ to represent the number of bits per symbol in the backward direction of each link (from the decoder to the encoders E_X and E_Y, respectively). The total number of bits per symbol pair sent in the forward direction is then given by

$$R_f(x^n, y^n|\mathcal{I}_n) = R_f^{(X)}(x^n, y^n|\mathcal{I}_n) + R_f^{(Y)}(x^n, y^n|\mathcal{I}_n)$$

Moreover, given x^n and y^n, let $Pe(\mathcal{I}_n|x^n, y^n)$ denote the indicator variable of error event of $\mathcal{I}_n$, defined as

$$Pe(\mathcal{I}_n|x^n, y^n) \triangleq \begin{cases} 0 & \text{if } \hat{x}^n = x^n, \hat{y}^n = y^n \\ 1 & \text{otherwise} \end{cases}$$

where $\hat{x}^n$ and $\hat{y}^n$ are the outputs of the decoder. For random IED, where $\mathcal{I}_n$ is chosen randomly from an IED ensemble $\mathbb{I}_n$, we further define

$$\begin{aligned} R_f^{(X)}(x^n, y^n|\mathbb{I}_n) &= \mathbf{E}[R_f^{(X)}(x^n, y^n|\mathcal{I}_n)] \\ R_f^{(Y)}(x^n, y^n|\mathbb{I}_n) &= \mathbf{E}[R_f^{(Y)}(x^n, y^n|\mathcal{I}_n)] \\ R_f(x^n, y^n|\mathbb{I}_n) &= \mathbf{E}[R_f(x^n, y^n|\mathcal{I}_n)] \\ R_b^{(X)}(x^n, y^n|\mathbb{I}_n) &= \mathbf{E}[R_b^{(X)}(x^n, y^n|\mathcal{I}_n)] \\ R_b^{(Y)}(x^n, y^n|\mathbb{I}_n) &= \mathbf{E}[R_b^{(Y)}(x^n, y^n|\mathcal{I}_n)] \\ Pe(\mathbb{I}_n|x^n, y^n) &= \mathbf{E}[Pe(\mathcal{I}_n|x^n, y^n)] \end{aligned}$$

where x^n and y^n are given, and $\mathbf{E}[\cdot]$ is the standard expectation operator, taken over $\mathbb{I}_n$.

III. Universal IED: Algorithms and Performance

Fix four classical lossless coding schemes $\mathcal{C}_n^{(X)}$, $\mathcal{C}_n^{(Y)}$, $\mathcal{C}_n^{(X|Y)}$, and $\mathcal{C}_n^{(Y|X)}$ with their respective normalized codeword length functions $l_n^{(X)}(\cdot)$, $l_n^{(Y)}(\cdot)$, $l_n^{(X|Y)}(\cdot|\cdot)$, and $l_n^{(Y|X)}(\cdot|\cdot)$. In this section, we first show how to convert these four classical lossless coding schemes into a universal random IED scheme for Figure 1, and then analyze the performance of the resulting IED scheme.

A. Universal Random IED Schemes

Let Δ_X and Δ_Y be two integer factors of $n\lceil\log|\mathcal{X}|\rceil$ and $n\lceil\log|\mathcal{Y}|\rceil$, respectively. Randomly generate two sequences of binary matrices $\{H_i^{(X)}\}_{i=0}^{\frac{n\lceil\log|\mathcal{X}|\rceil}{\Delta_X}}$ and $\{H_i^{(Y)}\}_{i=0}^{\frac{n\lceil\log|\mathcal{Y}|\rceil}{\Delta_Y}}$, where each $H_i^{(X)}$ is of size $\Delta_X \times n\lceil\log|\mathcal{X}|\rceil$, each $H_i^{(Y)}$ is of size $\Delta_Y \times n\lceil\log|\mathcal{Y}|\rceil$, and elements of $H_i^{(X)}$ and $H_i^{(Y)}$ are generated independently and uniformly over the binary alphabet $\{0,1\}$. With reference to Figure 1, the sequence $\{H_i^{(X)}\}_{i=0}^{\frac{n\lceil\log|\mathcal{X}|\rceil}{\Delta_X}}$ is revealed to the encoder E_X and joint decoder; likewise, the sequence $\{H_i^{(Y)}\}_{i=0}^{\frac{n\lceil\log|\mathcal{Y}|\rceil}{\Delta_Y}}$ is revealed to the encoder E_Y and joint decoder.

Based on $\{H_i^{(X)}\}_{i=0}^{\frac{n\lceil\log|\mathcal{X}|\rceil}{\Delta_X}}$ and $\{H_i^{(Y)}\}_{i=0}^{\frac{n\lceil\log|\mathcal{Y}|\rceil}{\Delta_Y}}$, we convert the four given classical lossless coding schemes $\mathcal{C}_n^{(X)}$, $\mathcal{C}_n^{(Y)}$, $\mathcal{C}_n^{(X|Y)}$, and $\mathcal{C}_n^{(Y|X)}$ into a universal random IED scheme $\mathcal{I}_n$ of order n, which encodes each pair of individual sequences x^n and y^n as follows:

- Set $j = 0$.
- The decoder sends bit 0 to both E_X and E_Y to initialize the transmission.
- Stage I [Transmission of Source Parity Check Bits from E_X and E_Y]:
  - Upon receiving bit 0, E_X transmits $S_j^{(X)} = H_j^{(X)} x_B^n$ to the decoder.
  - Upon receiving bit 0, E_Y transmits $S_j^{(Y)} = H_j^{(Y)} y_B^n$ to the decoder.
  - After receiving $S_j^{(X)}$ and $S_j^{(Y)}$, the decoder computes

$$\dot{x}^n = \operatorname*{argmin}_{v^n: H_i^{(X)} v_B^n = S_i^{(X)}, 0\le i\le j} l_n^{(X)}(v^n) \quad (1)$$

$$\dot{y}^n = \operatorname*{argmin}_{w^n: H_i^{(Y)} w_B^n = S_i^{(Y)}, 0\le i\le j} l_n^{(Y)}(w^n) \quad (2)$$

and check the following two conditions:

C1: $l_n^{(X)}(\dot{x}^n) \le \frac{(j-1)\Delta_X}{n}$ or $j \ge \frac{n\lceil\log|\mathcal{X}|\rceil}{\Delta_X}$;

C2: $l_n^{(Y)}(\dot{y}^n) \le \frac{(j-1)\Delta_Y}{n}$ or $j \ge \frac{n\lceil\log|\mathcal{Y}|\rceil}{\Delta_Y}$.

  - If both C1 and C2 are true, the decoder outputs $(\dot{x}^n, \dot{y}^n)$ as $(\hat{x}^n, \hat{y}^n)$ and sends bits 10 to E_X and E_Y; this IED scheme then terminates.
  - If only C1 is true, then the decoder outputs $\dot{x}^n$ as $\hat{x}^n$, sends bits 10 to E_X to terminate E_X, and then updates $\dot{y}^n$ into

$$\dot{y}^n = \operatorname*{argmin}_{w^n: H_i^{(Y)} w_B^n = S_i^{(Y)}, 0\le i\le j} l_n^{(Y|X)}(w^n|\hat{x}^n)\,. \quad (3)$$

If $l_n^{(Y|X)}(\dot{y}^n|\hat{x}^n) \le \frac{(j-1)\Delta_Y}{n}$, then the decoder outputs $\dot{y}^n$ as $\hat{y}^n$ and sends bits 10 to E_Y to terminate E_Y, and hence the whole IED process; otherwise, the decoder leaves $\hat{y}^n$ undecided and sends bit 0 to E_Y, and after increasing j by 1, the IED process moves to Stage III.
  - If only C2 is true, then the decoder outputs $\dot{y}^n$ as $\hat{y}^n$, sends bits 10 to E_Y to terminate E_Y, and then updates $\dot{x}^n$ into

$$\dot{x}^n = \operatorname*{argmin}_{v^n: H_i^{(X)} v_B^n = S_i^{(X)}, 0\le i\le j} l_n^{(X|Y)}(v^n|\hat{y}^n)\,. \quad (4)$$

If $l_n^{(X|Y)}(\dot{x}^n|\hat{y}^n) \le \frac{(j-1)\Delta_X}{n}$, then the decoder outputs $\dot{x}^n$ as $\hat{x}^n$ and sends bits 10 to E_X to terminates E_X and hence the whole IED process; otherwise, the decoder leaves $\hat{x}^n$ undecided and sends bit 0 to E_X, and after increasing j by 1, the IED process moves to Stage IV.
  - If neither C1 nor C2 is true, the decoder updates $(\dot{x}^n, \dot{y}^n)$ into

$$(\dot{x}^n, \dot{y}^n) = \operatorname*{argmin}_{(v^n, w^n)\in\Lambda_j} L_n^{(m)}(v^n, w^n) \quad (5)$$

where Λ_j is a set of (v^n, w^n) satisfying

$$H_i^{(X)} v_B^n = S_i^{(X)}, H_i^{(Y)} w_B^n = S_i^{(Y)}, \text{ for } 0 \le i \le j$$

and check the following condition

C3: $L_n^{(m)}(\dot{x}^n, \dot{y}^n) \le \frac{(j-1)(\Delta_X+\Delta_Y)}{n}$

  - If C3 is true, then the decoder records $(\dot{x}^n, \dot{y}^n)$ as a tentative estimation $(\tilde{x}^n, \tilde{y}^n)$ of (x^n, y^n) and sends bits 11 followed by $\tilde{S}^{(X)} = \tilde{H}^{(X)} \tilde{x}_B^n$ to E_X, and bits 11 followed by $\tilde{S}^{(Y)} = \tilde{H}^{(Y)} \tilde{y}_B^n$ to E_Y, where $\tilde{H}^{(X)} = H_{j+1}^{(X)}$, $\tilde{H}^{(Y)} = H_{j+1}^{(Y)}$, and hereafter the IED process moves to Stage II.
  - If C3 is not true, the decoder sends bit 0 to E_X and E_Y; increase j by 1, and the IED process remains at Stage I.
- Stage II [Transmission of Parity Comparison Bit from E_X and E_Y]:

– Upon receiving bits 11 and $\tilde{S}^{(X)}$, E_X computes $\tilde{H}^{(X)}x_B^n$; the encoder E_X transmits bit 1 to the decoder if $\tilde{S}^{(X)} = \tilde{H}^{(X)}x_B^n$, and bit 0 otherwise.
– Upon receiving bits 11 and $\tilde{S}^{(Y)}$, E_Y computes $\tilde{H}^{(Y)}y_B^n$; the encoder E_Y transmits bit 1 to the decoder if $\tilde{S}^{(Y)} = \tilde{H}^{(Y)}y_B^n$, and bit 0 otherwise.
– If the decoder receives bit 1 from E_X and bit 0 from E_Y, it outputs $\tilde{x}^n$ as $\hat{x}^n$, sends bits 10 to E_X to terminate E_X, and then updates $\dot{y}^n$ according to (3). If $l_n^{(Y|X)}(\dot{y}^n|\hat{x}^n) \leq \frac{(j-1)\Delta_Y}{n}$, then the decoder further outputs $\dot{y}^n$ as $\hat{y}^n$, and sends bits 10 to E_Y to terminate E_Y and hence the whole IED process; otherwise, the decoder leaves $\hat{y}^n$ undecided and sends bit 0 to E_Y, and after increasing j by 1, the IED process moves to Stage III.
– If the decoder receives bit 0 from E_X and bit 1 from E_Y, it outputs $\tilde{y}^n$ as $\hat{y}^n$, sends bits 10 to E_Y to terminate E_Y, and then updates $\dot{x}^n$ according to (4). if $l_n^{(X|Y)}(\dot{x}^n|\hat{y}^n) \leq \frac{(j-1)\Delta_X}{n}$, then the decoder further outputs $\dot{x}^n$ as $\hat{x}^n$ and sends bits 10 to E_X to terminate E_X and hence the whole IED process; otherwise, the decoder leaves $\hat{x}^n$ undecided and sends bit 0 to E_X, and after increasing j by 1, the IED process moves to Stage IV.
– If the decoder receives bit 0 or bit 1 from both E_X and E_Y, it outputs $(\tilde{x}^n, \tilde{y}^n)$ as $(\hat{x}^n, \hat{y}^n)$ and sends bits 10 to E_X and E_Y to terminate both E_X and E_Y, and hence the whole IED process.

- Stage III [Transmission of Source Parity Check Bits from E_Y Only]:
  – Upon receiving bit 0, E_Y transmits $S_j^{(Y)} = H_j^{(Y)}y_B^n$ to the decoder.
  – After receiving $S_j^{(Y)}$, the decoder computes (2), and check the condition C2.
  – If C2 is true, then the decoder outputs $\dot{y}^n$ as $\hat{y}^n$ and sends bits 10 to E_Y to terminate E_Y and hence the whole IED process.
  – If C2 is not true, the decoder updates $\dot{y}^n$ according to (3). If $l_n^{(Y|X)}(\dot{y}^n|\hat{x}^n) \leq \frac{(j-1)\Delta_Y}{n}$, then the decoder outputs $\dot{y}^n$ as $\hat{y}^n$ and sends bits 10 to E_Y to terminate E_Y and hence the whole IED process; otherwise, the decoder leaves $\hat{y}^n$ undecided and sends bit 0 to E_Y, and after increasing j by 1, the IED process remains at Stage III.
- Stage IV [Transmission of Source Parity Check Bits from E_X Only]:
  – Upon receiving bit 0, E_X transmits $S_j^{(X)} = H_j^{(X)}x_B^n$ to the decoder;
  – After receiving $S_j^{(Y)}$, the decoder computes (1), and check the condition C1.
  – If C1 is true, then the decoder outputs $\dot{x}^n$ as $\hat{x}^n$ and sends bits 10 to E_X to terminate E_X and hence the whole IED process.
  – If C1 is not true, then decoder updates $\dot{x}^n$ according to (4). If $l_n^{(X|Y)}(\dot{x}^n|\hat{y}^n) \leq \frac{(j-1)\Delta_X}{n}$, then the decoder further outputs $\dot{x}^n$ as $\hat{x}^n$ and sends bits 10 to E_X to terminate E_X and hence the whole IED process; otherwise, the decoder leaves $\hat{x}^n$ undecided and sends bit 0 to E_X, and after increasing j by 1, the IED process remains at Stage IV.

Remark 1: Note that the universal random IED scheme $\mathcal{I}_n$ of order n constructed above depends on the four classical lossless coding schemes $\mathcal{C}_n^{(X)}$, $\mathcal{C}_n^{(Y)}$, $\mathcal{C}_n^{(X|Y)}$, and $\mathcal{C}_n^{(Y|X)}$ only through their respective normalized codeword length functions.

B. Performance

The following theorem shows that for each and every pair of individual sequences x^n and y^n, the average number of total bits per symbol pair to be exchanged between the joint decoder and two separate encoders of the universal random IED scheme $\mathcal{I}_n$ is roughly upper bounded by the total number of bits per symbol pair in the joint encoding and decoding of x^n and y^n (via the encoding of x^n plus the conditional encoding of y^n given x^n or the encoding of y^n plus the conditional encoding of x^n given y^n).

Theorem 1: For any sequence (x^n, y^n),

$$R_f^{(X)}(x^n, y^n|\mathbb{I}_n) \leq l_n^{(X)}(x^n) + \frac{3\Delta_X + 1}{n} \tag{6}$$

$$R_f^{(Y)}(x^n, y^n|\mathbb{I}_n) \leq l_n^{(Y)}(y^n) + \frac{3\Delta_Y + 1}{n} \tag{7}$$

$$\begin{aligned} R_f(x^n, y^n|\mathbb{I}_n) \leq\ & L_n^{(M)}(x^n, y^n) + \frac{3(\Delta_X + \Delta_Y) + 2}{n} \\ & + Q_{\Delta_X,\Delta_Y}\left(\lceil \log|\mathcal{Y}|\rceil + \lceil \log|\mathcal{X}|\rceil\right) \end{aligned} \tag{8}$$

$$R_b^{(X)}(x^n, y^n|\mathbb{I}_n) \leq \frac{\lceil \log|\mathcal{X}|\rceil}{\Delta_X} + \frac{\Delta_X + 5}{n} \tag{9}$$

$$R_b^{(Y)}(x^n, y^n|\mathbb{I}_n) \leq \frac{\lceil \log|\mathcal{Y}|\rceil}{\Delta_Y} + \frac{\Delta_Y + 5}{n} \tag{10}$$

$$Pe(\mathbb{I}_n|x^n, y^n) \leq Q_{\Delta_X,\Delta_Y} \tag{11}$$

where

$$\begin{aligned} & Q_{\Delta_X,\Delta_Y} \\ \triangleq\ & 2^{-\Delta_X+1+\log\left(\frac{n\lceil \log|\mathcal{X}|\rceil}{\Delta_X}+1\right)} \\ & +2^{-\Delta_Y+1+\log\left(\frac{n\lceil \log|\mathcal{Y}|\rceil}{\Delta_Y}+1\right)} \\ & +2^{-\Delta_X-\Delta_Y+1+\log\left(\min\{\frac{n\lceil \log\{|\mathcal{X}_n|\}\rceil}{\Delta_X}, \frac{n\lceil \log|\mathcal{Y}_n|\rceil}{\Delta_Y}\}\right)} \end{aligned}$$

A sketch of the proof of Theorem 1 is provided in Appendix A.

In Theorem 1, we mainly focus on the bound $R_f(x^n, y^n|\mathcal{I}_n)$. In fact, if the ratio of Δ_X and Δ_Y is given, under some mild conditions on the code length functions, it is possible to derive tighter bounds for $R_f^{(X)}(x^n, y^n|\mathcal{I}_n)$ and $R_f^{(Y)}(x^n, y^n|\mathcal{I}_n)$ separately while maintaining the same bound on $R_f(x^n, y^n|\mathcal{I}_n)$ as in Theorem 1.

In the following discussion, we assume

$$l_n^{(X|Y)}(x^n|y^n) \leq l_n^{(X)}(x^n) \tag{12}$$

$$l_n^{(Y|X)}(y^n|x^n) \leq l_n^{(Y)}(y^n) \tag{13}$$

The assumptions can be interpreted as that side information can not make the coding performance worse.

Theorem 2: Under the assumptions of (12) and (13), each of the following hold.

(i) If

$$\frac{\Delta_Y}{\Delta_X} \geq \frac{l_n^{(Y)}(y^n)}{l_n^{(X|Y)}(x^n|y^n)}$$

then

$$\begin{aligned} R_f^{(X)}(x^n, y^n|\mathcal{I}_n) &\leq l_n^{(X|Y)}(x^n|y^n) + \frac{3\Delta_X + 1}{n} \\ &+ Q_{\Delta_X, \Delta_Y} \lceil \log |\mathcal{X}| \rceil \end{aligned}$$

and

$$R_f^{(Y)}(x^n, y^n|\mathbb{I}_n) \leq l_n^{(Y)}(y^n) + \frac{3\Delta_Y + 1}{n}$$

(ii) If

$$\frac{\Delta_Y}{\Delta_X} \leq \frac{l_n^{(Y|X)}(y^n|x^n)}{l_n^{(X)}(x^n)}$$

then

$$\begin{aligned} R_f^{(Y)}(x^n, y^n|\mathcal{I}_n) &\leq l_n^{(Y|X)}(y^n|x^n) + \frac{3\Delta_Y + 1}{n} \\ &+ Q_{\Delta_X, \Delta_Y} \lceil \log |\mathcal{Y}| \rceil \end{aligned}$$

and

$$R_f^{(X)}(x^n, y^n|\mathbb{I}_n) \leq l_n^{(X)}(x^n) + \frac{3\Delta_X + 1}{n}$$

(iii) If

$$\frac{l_n^{(Y|X)}(y^n|x^n)}{l_n^{(X)}(x^n)} < \frac{\Delta_Y}{\Delta_X} < \frac{l_n^{(Y)}(y^n)}{l_n^{(X|Y)}(x^n|y^n)}$$

then

$$\begin{aligned} & R_f^{(Y)}(x^n, y^n|\mathcal{I}_n) \leq \\ & \quad \frac{\Delta_Y (l_n^{(X)}(x^n) + l_n^{(Y|X)}(y^n|x^n))}{\Delta_X + \Delta_Y} \\ & \quad + \frac{3\Delta_Y + 1}{n} + Q_{\Delta_X, \Delta_Y} \lceil \log |\mathcal{Y}| \rceil \end{aligned}$$

and

$$\begin{aligned} & R_f^{(X)}(x^n, y^n|\mathcal{I}_n) \leq \\ & \quad \frac{\Delta_X (l_n^{(Y)}(y^n) + l_n^{(X|Y)}(x^n|y^n))}{\Delta_X + \Delta_Y} \\ & \quad + \frac{3\Delta_X + 1}{n} + Q_{\Delta_X, \Delta_Y} \lceil \log |\mathcal{X}| \rceil \end{aligned}$$

The proof of Theorem 2, provided in the full paper [6], is the refinement of that of Theorem 1, where the ratio of Δ_X and Δ_Y is taken into account.

Theorem 2 can be interpreted graphically, as shown in Figure 2, where we assume that

$$l_n^{(X)}(x^n) + l_n^{(Y|X)}(y^n|x^n) = l_n^{(Y)}(y^n) + l_n^{(X|Y)}(x^n|y^n)$$

for the simplification of the discussion. As can been in this figure, different values of the ratio Δ_X / Δ_Y will result in different values of $R_f^{(X)}(x^n, y^n|\mathcal{I}_n)$ and $R_f^{(Y)}(x^n, y^n|\mathcal{I}_n)$, while $R_f(x^n, y^n|\mathcal{I}_n)$ remains the same.

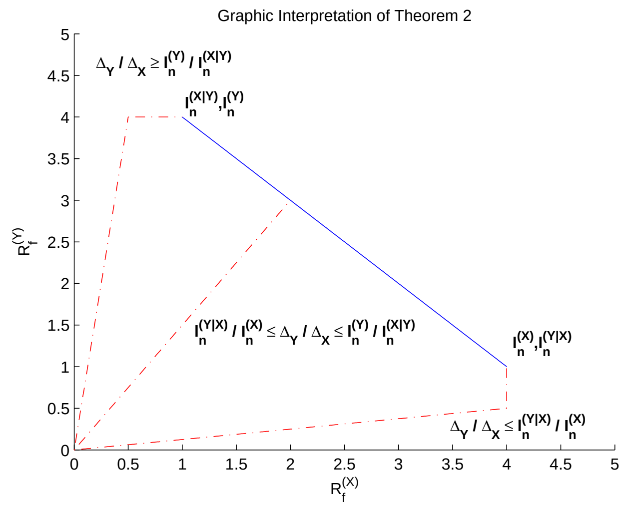

Fig. 2. Graphic Interpretation of Theorem 2

IV. Conclusion

In this paper, Interactive Encoding and Decoding (IED) for distributed lossless coding of individual sequences is considered. Coupled with classical source codes, we propose IED schemes which can be applied to any individual sequence pair. While without interactions, joint decoder for distributed lossless coding of individual sequences does not improve the coding performance compared to the system in which each sequence is decoded separately, we show that our schemes can approach the same sum rate performance as the system of jointly encoding and decoding sequence pairs. Moreover, our analysis shows that we can adjust the rate of each links while maintaining the sum rate by choosing the proper parameters for the schemes.

Appendix A
Sketch of the Proof of Theorem 1

Given individual sequences x^n and y^n, let $j_X(x^n, y^n|\mathcal{I}_n)$ and $j_Y(x^n, y^n|\mathcal{I}_n)$ be the numbers of times when E_X and E_Y transmit source parity check bits to the decoder respectively. From the description of $\mathcal{I}_n$, it is not hard to see that $j_X(x^n, y^n|\mathcal{I}_n)$ is always upper-bounded by $\frac{n\lceil \log |\mathcal{X}| \rceil}{\Delta_X} + 1$, and

$$R_f^{(X)}(x^n, y^n|\mathcal{I}_n) \leq \frac{j_X(x^n, y^n|\mathcal{I}_n)\Delta_X}{n} + \frac{1}{n} \quad \text{(A.1)}$$

$$R_b^{(X)}(x^n, y^n|\mathcal{I}_n) \leq \frac{j_X(x^n, y^n|\mathcal{I}_n) + 4}{n} + \frac{\Delta_X}{n} \quad \text{(A.2)}$$

From this, (9) follows.

To complete the proof of (6), observe that upon receiving the $(j_X(x^n, y^n|\mathcal{I}_n) - 1)$th packet of source parity check bits from E_X,

$$\begin{aligned} l_n^{(X)}(x^n) &\geq \min_{v^n : H_i^{(X)} v_B^n = S_i^{(X)}, 0 \leq i \leq j_X(x^n, y^n|\mathcal{I}_n) - 2} l_n^{(X)}(v^n) \\ &\geq \frac{(j_X(x^n, y^n|\mathcal{I}_n) - 3)\Delta_X}{n} \end{aligned} \quad \text{(A.3)}$$

no matter what stage the IED process of $\mathcal{I}_n$ is on at that point. Plugging (A.3) into (A.1) yields (6), while (7) and (10) can be proved in the same manner. Therefore, the proof of Theorem 1 is finished once (8) and (11) are proved.

From the description of $\mathcal{I}_n$ again, one can show that if

$$Pe\{\mathcal{I}_n|x^n, y^n\} = 1$$

then at least one of the following events happens:

F_1: For some j, $0 \le j < \frac{n\lceil \log\{|\mathcal{X}_n|\}\rceil}{\Delta_X}$

$$\begin{aligned} \exists \hat{x}^n \neq x^n, H_i^{(X)}\hat{x}_B^n &= H_i^{(X)}x_B^n \text{ for } 0 \le i \le j, \text{ and} \\ l_n^{(X)}(\hat{x}^n) &\le \frac{(j-1)\Delta_X}{n}. \end{aligned}$$

F_2: For some j, $0 \le j < \frac{n\lceil \log|\mathcal{Y}_n|\rceil}{\Delta_Y}$,

$$\begin{aligned} \exists \hat{y}^n \neq y^n, H_i^{(Y)}\hat{y}_B^n &= H_i^{(Y)}y_B^n \text{ for } 0 \le i \le j, \text{ and} \\ l_n^{(Y)}(\hat{y}^n) &\le \frac{(j-1)\Delta_Y}{n}. \end{aligned}$$

F_3: For $j = \frac{n\lceil \log\{|\mathcal{X}_n|\}\rceil}{\Delta_X}$

$$\exists \hat{x}^n \neq x^n, H_i^{(X)}\hat{x}_B^n = H_i^{(X)}x_B^n \text{ for } 0 \le i \le j.$$

F_4: For $j = \frac{n\lceil \log|\mathcal{Y}_n|\rceil}{\Delta_Y}$,

$$\exists \hat{y}^n \neq y^n, H_i^{(Y)}\hat{y}_B^n = H_i^{(Y)}y_B^n \text{ for } 0 \le i \le j.$$

F_5: For some j, $0 \le j < \min\{\frac{n\lceil \log\{|\mathcal{X}_n|\}\rceil}{\Delta_X}, \frac{n\lceil \log|\mathcal{Y}_n|\rceil}{\Delta_Y}\}$, $\exists \hat{x}^n \neq x^n, \hat{y}^n \neq y^n$,

$$\begin{aligned} H_i^{(X)}\hat{x}_B^n &= H_i^{(X)}x_B^n, \\ H_i^{(Y)}\hat{y}_B^n &= H_i^{(Y)}y_B^n \end{aligned}$$

for $0 \le i \le j$, and

$$L_n^{(m)}(\hat{x}^n, \hat{y}^n) \le \frac{(j-1)(\Delta_X + \Delta_Y)}{n}.$$

F_6: $\tilde{x}^n \neq x^n$, and

$$H_{j+1}^{(X)}\tilde{x}_B^n = H_{j+1}^{(X)}x_B^n$$

where j is the index of the interaction immediately before the IED scheme enters stage II.

F_7: $\tilde{y}^n \neq y^n$,

$$H_{j+1}^{(Y)}\tilde{y}_B^n = H_{j+1}^{(Y)}y_B^n$$

where j is the index of the interaction immediately before the IED scheme enters stage II.

F_8: For some j, $0 \le j < \frac{n\lceil \log\{|\mathcal{X}_n|\}\rceil}{\Delta_X}$

$$\begin{aligned} \exists \hat{x}^n \neq x^n, H_i^{(X)}\hat{x}_B^n &= H_i^{(X)}x_B^n \text{ for } 0 \le i \le j, \text{ and} \\ l_n^{(X|Y)}(\hat{x}^n|y^n) &\le \frac{(j-1)\Delta_X}{n}. \end{aligned}$$

F_9: For some j, $0 \le j < \frac{n\lceil \log|\mathcal{Y}_n|\rceil}{\Delta_Y}$,

$$\begin{aligned} \exists \hat{y}^n \neq y^n, H_i^{(Y)}\hat{y}_B^n &= H_i^{(Y)}y_B^n \text{ for } 0 \le i \le j, \text{ and} \\ l_n^{(Y|X)}(\hat{y}^n|x^n) &\le \frac{(j-1)\Delta_Y}{n}. \end{aligned}$$

By the union bound,

$$\begin{aligned} Pe(\mathbb{I}_n|x^n, y^n) &= \mathbf{E}[Pe(\mathcal{I}_n|x^n, y^n)] \\ &\le \sum_{i=1}^{9} \Pr\{F_i\} \end{aligned}$$

Therefore, $Pe(\mathbb{I}_n|x^n, y^n)$ will be bounded if we give the bound on the probability of each of F_i.

For F_1, by the union bound again

$$\begin{aligned} \Pr\{F_1\} &\le \sum_j |A_X^{\frac{(j-1)\Delta_X}{n}}| 2^{-j\Delta_X} \\ &\le 2^{-\Delta_X + \log\left(\frac{n\lceil \log|\mathcal{X}|\rceil}{\Delta_X}\right)} \end{aligned}$$

In the similar manner, we can derive the bounds for F_2, F_3, F_4, F_5, F_8, and F_9,

$$\begin{aligned} \Pr\{F_2\} &\le 2^{-\Delta_Y + \log\left(\frac{n\lceil \log|\mathcal{Y}|\rceil}{\Delta_Y}\right)} \\ \Pr\{F_3\} &\le 2^{-\Delta_X} \\ \Pr\{F_4\} &\le 2^{-\Delta_Y} \\ \Pr\{F_5\} &\le 2^{-\Delta_X - \Delta_Y + 1 + \log\left(\min\{\frac{n\lceil \log\{|\mathcal{X}_n|\}\rceil}{\Delta_X}, \frac{n\lceil \log|\mathcal{Y}_n|\rceil}{\Delta_Y}\}\right)} \\ \Pr\{F_8\} &\le 2^{-\Delta_X + \log\left(\frac{n\lceil \log|\mathcal{X}|\rceil}{\Delta_X}\right)} \\ \Pr\{F_9\} &\le 2^{-\Delta_Y + \log\left(\frac{n\lceil \log|\mathcal{Y}|\rceil}{\Delta_Y}\right)} \end{aligned}$$

Now for F_6, both j and $\tilde{x}^n$ are random variables. Since $\tilde{x}^n$ (if it exists) is determined by $\{H_i^{(X)}\}_{i=0}^j$ and $\{H_i^{(Y)}\}_{i=0}^j$, it is independent with $H_{j+1}^{(X)}$. Therefore, we have

$$\begin{aligned} \Pr\{F_6\} &\le \Pr\{H_i^{(X)}\tilde{x}^n = H_i^{(X)}x^n | \tilde{x}^n \neq x^n\} \\ &= 2^{-\Delta_X} \end{aligned}$$

and similarly

$$\Pr\{F_7\} \le 2^{-\Delta_Y}$$

All in all, we have

$$\begin{aligned} & Pe(\mathbb{I}_n|x^n, y^n) \\ \le\ & 2^{-\Delta_X + 1 + \log\left(\frac{n\lceil \log|\mathcal{X}|\rceil}{\Delta_X} + 1\right)} \\ & + 2^{-\Delta_Y + 1 + \log\left(\frac{n\lceil \log|\mathcal{Y}|\rceil}{\Delta_Y} + 1\right)} \\ & + 2^{-\Delta_X - \Delta_Y + 1 + \log\left(\min\{\frac{n\lceil \log\{|\mathcal{X}_n|\}\rceil}{\Delta_X}, \frac{n\lceil \log|\mathcal{Y}_n|\rceil}{\Delta_Y}\}\right)} \end{aligned}$$

and (11) has been proved.

Towards proving (8), given x^n and y^n, divide $\mathbb{I}_n$ into three sub-ensembles $\mathbb{I}_n^{(\mathrm{I,II})}$, $\mathbb{I}_n^{(\mathrm{III})}$, $\mathbb{I}_n^{(\mathrm{IV})}$ for which the whole IED process of $\mathcal{I}_n$ terminates at Stages I or II, Stage III, and Stage IV respectively. Then

$$\begin{aligned} R_f(x^n, y^n|\mathbb{I}_n) &= \Pr\{\mathcal{I}_n \in \mathbb{I}_n^{(\mathrm{I,II})}\} R_f(x^n, y^n|\mathbb{I}_n^{(\mathrm{I,II})}) \\ &\quad + \Pr\{\mathcal{I}_n \in \mathbb{I}_n^{(\mathrm{III})}\} R_f(x^n, y^n|\mathbb{I}_n^{(\mathrm{III})}) \\ &\quad + \Pr\{\mathcal{I}_n \in \mathbb{I}_n^{(\mathrm{IV})}\} R_f(x^n, y^n|\mathbb{I}_n^{(\mathrm{IV})}) \end{aligned}$$

- For $\mathcal{I}_n \in \mathbb{I}_n^{(\text{I,II})}$, let $j(x^n, y^n|\mathcal{I}_n)$ represent the j when the whole IED process of $\mathcal{I}_n$ terminates. Note that in this case

$$j_X(x^n, y^n|\mathcal{I}_n) = j_Y(x^n, y^n|\mathcal{I}_n) = j(x^n, y^n|\mathcal{I}_n) + 1$$

and

$$\begin{aligned} R_f(x^n, y^n|\mathcal{I}_n) &\leq \frac{(j(x^n, y^n|\mathcal{I}_n) + 1)(\Delta_X + \Delta_Y)}{n} \\ &\quad + \frac{2}{n} \end{aligned}$$

On the other hand, at the $j(x^n, y^n|\mathcal{I}_n) - 1$ round of interaction,

$$\begin{aligned} & L_n^{(m)}(x^n, y^n) \\ \geq & \min_{(v^n, w^n) \in \Lambda_{j(x^n, y^n|\mathcal{I}_n)-1}} L_n^{(m)}(v^n, w^n) \\ \geq & \frac{(j(x^n, y^n|\mathcal{I}_n) - 2)(\Delta_X + \Delta_Y)}{n} \end{aligned}$$

where $\Lambda_{j(x^n, y^n|\mathcal{I}_n)-1}$ follows the definition in the description of IED schemes. Therefore,

$$\begin{aligned} R_f(x^n, y^n|\mathcal{I}_n) &\leq L_n^{(m)}(x^n, y^n) + \frac{3(\Delta_X + \Delta_Y)}{n} \\ &\quad + \frac{2}{n} \end{aligned} \tag{A.4}$$

- For $\mathcal{I}_n \in \mathbb{I}_n^{(\text{III})}$, upon receiving $(j_Y(x^n, y^n|\mathcal{I}_n) - 1)$th packet of source parity check bits from E_Y,

$$\begin{aligned} & l_{Y|X}(y^n|\hat{x}^n) \\ \geq & \min_{w^n : H_i^{(Y)} w_B^n = S_i^{(Y)}, 0 \leq i \leq j_Y(x^n, y^n|\mathcal{I}_n)-2} l_n^{(Y|X)}(w^n|\hat{x}^n) \\ \geq & \frac{(j_Y(x^n, y^n|\mathcal{I}_n) - 3)\Delta_Y}{n} \end{aligned}$$

therefore,

$$R_f^{(Y)}(x^n, y^n|\mathcal{I}_n) \leq l_n^{(Y|X)}(y^n|\hat{x}^n) + \frac{3\Delta_Y}{n} + \frac{1}{n}$$

Now for $R_f^{(X)}(x^n, y^n|\mathcal{I}_n)$, we have

$$R_f^{(X)}(x^n, y^n|\mathcal{I}_n) \leq l_n^{(X)}(x^n) + \frac{3\Delta_X}{n} + \frac{1}{n}$$

therefore,

$$\begin{aligned} R_f(x^n, y^n|\mathcal{I}_n) &\leq l_n^{(X)}(x^n) + l_n^{(Y|X)}(y^n|\hat{x}^n) \\ &\quad + \frac{3(\Delta_X + \Delta_Y)}{n} + \frac{2}{n} \end{aligned}$$

In the meantime, we have

$$j_Y(x^n, y^n|\mathcal{I}_n) \leq \frac{\lceil \log |\mathcal{Y}| \rceil}{\Delta_Y} + 1$$

which implies that

$$\begin{aligned} R_f(x^n, y^n|\mathcal{I}_n) &\leq l_n^{(X)}(x^n) + \frac{3(\Delta_X + \Delta_Y)}{n} + \frac{2}{n} \\ &\quad + \min\{l_n^{(Y|X)}(y^n|\hat{x}^n), \lceil \log |\mathcal{Y}| \rceil\} \end{aligned}$$

- Similar for the case $\mathcal{I}_n \in \mathbb{I}_n^{(\text{IV})}$,

$$\begin{aligned} R_f(x^n, y^n|\mathcal{I}_n) &\leq l_n^{(Y)}(y^n) + \frac{3(\Delta_X + \Delta_Y)}{n} + \frac{2}{n} \\ &\quad + \min\{l_n^{(X|Y)}(x^n|\hat{y}^n), \lceil \log |\mathcal{X}| \rceil\} \end{aligned}$$

In total, we have

$$\begin{aligned} & R_f(x^n, y^n|\mathbb{I}_n) \\ \leq & L_n^{(M)}(x^n, y^n) + \frac{3(\Delta_X + \Delta_Y)}{n} \\ & + \Pr\{\hat{x}^n \neq x^n\} \lceil \log |\mathcal{Y}| \rceil \\ & + \Pr\{\hat{y}^n \neq y^n\} \lceil \log |\mathcal{X}| \rceil \end{aligned}$$

Then (8) is proved by noting that $\Pr\{\hat{x}^n \neq x^n\}$ and $\Pr\{\hat{y}^n \neq y^n\}$ are bounded by $Pe(\mathbb{I}_n|x^n, y^n)$.

References

[1] D. Slepian and J. K. Wolf, "Noiseless coding of correlated information sources," *IEEE Trans. Inf. Theory*, vol. IT-19, pp. 471–480, July 1973.

[2] T. M. Cover, "A proof of the data compression theorem of slepian and wolf for ergodic sources," *IEEE Trans. Inf. Theory*, vol. IT-19, pp. 226–228, Mar. 1975.

[3] R. F. Ahlswede and J. Körner, "Source coding with side information and a converse for degraded broadcast channels," *IEEE Trans. Inf. Theory*, vol. IT-21, 1975.

[4] E.-H. Yang and D.-K. He, "On interactive encoding and decoding for lossless source coding with decoder only side information," in *Proc. of ISIT'08*, July 2008, pp. 419–423.

[5] ——, "Interactive encoding and decoding for one way learning: Near lossless recovery with side information at the decoder," accepted for publication in IEEE Trans on Information Theory, 2009.

[6] E.-H. Yang and J. Meng, "Interactive encoding and decoding for distributed lossless coding of individual sequences," in preparation for IEEE Trans. Inform. Theory.