Adaptive Spectrum Sensing

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***Abstract*— A challenge met in realizing spectrum sensing for cognitive radio is how to identify occupied channels quickly, thus reducing the time required to identify unoccupied channels that can be utilized by the cognitive radio. The spectrum sensor makes a series of measurements and then utilizes those measurements to determine if the channel is occupied or not. An adaptive spectrum sensing scheme is proposed, which exploits the simple fact that for typical spectrum sensing systems the accumulation of channel statistics is a non-decreasing process. The adaptive sensing scheme is a variable-length decision procedure, having a hard limit on the maximum number of observations, and yielding the same performance as the optimal fixed-sample-size Neyman-Pearson detection. The behavior of the expected number of observations is investigated, and is compared with the sequential probability ratio tests.**

I. Introduction

One prerequisite for secondary users to exploit the unused spectrum in a cognitive radio network is to reliably detect whether a primary user is operating on the same frequency band, also know as spectrum sensing. Such requirement protects the primary user from experiencing harmful interference caused by the secondary users. Recently, the Federal Communications Commission (FCC) has issued the regulatory rules [3] for cognitive radio use of the TV white space spectrum. It requires that the cognitive devices who intend to operate in the TV white space must reliably detect TV (including ATSC and NTSC) as well as wireless microphone signals at a power level of -114 dBm. Moreover, spectrum sensing is required not only at the device start-up, but also periodically during the normal operation.

These requirements pose significant challenges to the design of cognitive radio devices [4]. For example, the device needs to cease transmission during the sensing times to avoid causing interference to the primary user detection. Such quiet times result in a reduction of data throughput, increase of latency, and sometimes loss of data, and hence should be kept minimum. On the other hand, the sensing performance is proportional to the length of measurement, and due to the stringent requirement of FCC, a relatively long sensing time is inevitable. This inherent conflict poses challenges to the system design.

The purpose of this work is to study how to achieve good spectrum sensing performance while reducing the quieting times. Traditional spectrum sensor makes a series of measurements and at the end of all measurement determines if the channel is occupied or not. The proposed adaptive spectrum sensing scheme exploits the simple fact that for typical spectrum sensing systems the accumulation of channel statistics is a non-decreasing process. Hence in the Neyman-Pearson detection, whenever the accumulated statistic exceeds the threshold a declaration can be made without making more observations. Such adaptive sensing can be proved to yield the same detection performance as the fixed-sample-size detection while reducing the number of observations.

The rest of the paper is organized as follows. Section II presents the system model and describes the problem. Section III briefly describes the conventional Neyman-Pearson type spectrum sensing. Section IV gives the proposed adaptive sensing algorithm and analyzes the detection performance as well as the expected number of observations. Section V uses an example to illustrate the benefit of the proposed algorithm. Section VI compares the adaptive sensing algorithm to the sequential probability ratio test. Section VII focuses on whether the adaptive algorithm minimizes the expected number of observations among all length-N detection procedures and proves a special case of N=2. Finally Section VIII concludes the paper.

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II. Problem Description

In TV white space applications, the FCC rules [3] require TV band devices (TVBD) to reliably sense incumbent TV signals. The sensing sensitivity requirement is that TV signals should be reliably detected at -114 dBm across a 6 MHz TV channel assuming a 0dBi receive antenna. This translates to a very low signal-to-noise ratio (SNR) of around -15 dB or lower depending on the receiver noise figure. If the receiver uses internal phone form-factor antennas (with a negative antenna gain typically about -5 dBi), the actual operating SNR at baseband processing for personal/portable devices can be as low as -20 dB.

In order to reliably detect TV band incumbent signals at this very low SNR, the TVBD needs to listen to the medium for a relatively long time in order to capture and analyze the received signals. The long dwelling time is needed to achieve processing gain to increase the SNR of the signals. The TVBD needs to be quiet (stop transmitting) during this signal collection phase to avoid any leakage from the transmitter power to the collected signals. We denote the time used by the device to collect signals as quiet time.

However, at any location not all of the TV channels are received at this very low power of -114dBm. TV signals received power on different channels can span a relatively large range. Such higher power in received TV signals may require

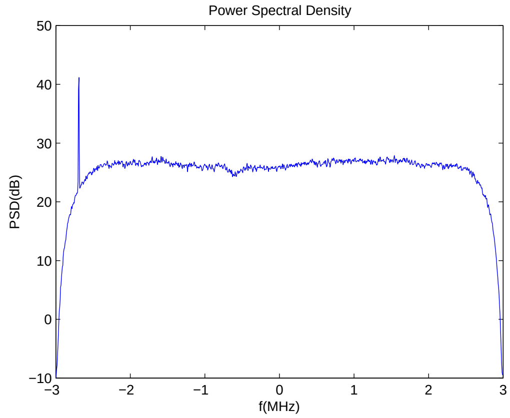

Fig. 1. Received PSD of a typical ATSC signal. The scale of the y-axis is not normalized. The pilot carrier exhibits like a narrow spike near the lower edge of the TV channel.

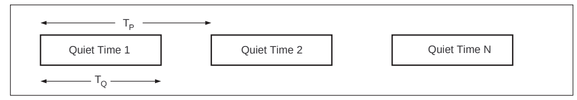

Fig. 2. Quiet Time Design.

less processing gain than the signals at -114dBm and hence may require less quiet time to achieve the same performance.

Although the proposed algorithm could be applied to TV sensing in general (ATSC and NTSC), we will focus on ATSC in describing the problem. The digital TV in the United States follows the ATSC standard. TV programs are modulated using 8-level vestigial sideband modulation (8VSB) and the modulated signal occupies almost the entire 6 MHz TV channel uniformly, with a pilot carrier which contains approximately 7% of the total signal power and is located at approximately 310 kHz above the lower edge of the channel. Figure 1 displays the received power spectral density (PSD) of a typical ATSC signal.

The key observation is that, ATSC signals statistically do not resemble white Gaussian noise, rather they exhibit narrowband features through the ATSC pilot. Furthermore, the frequency location of this narrowband feature is fixed, and the "local" SNR in their vicinity can be boosted through appropriate filtering. The spectrum sensing problem hence can be posed as a binary hypothesis test between pure noise and a carrier with unknown phase in noise.

Since sensing involves listening to the channel and trying to detect very weak signals, the white space device needs to stop transmitting during sensing time. The longer the quiet time the more processing gain could be achieved. The tradeoff, however, is that quieting impacts two important performance metrics, namely, throughput and latency. For design purposes, it is preferred to break the total required quiet time into a sequence of disjoint quiet periods. Throughput is affected by the total aggregate quiet time required , while latency is determined by the duration of a single quiet time, and so the length of one quiet period determines latency. Figure 2 depicts quiet time design for sensing, in which T_p denotes quiet time period and T_Q denotes quiet time duration.

Assume N total quiet times are used to decide on sensing outcome. Furthermore assume the statistics calculated from all quiet times are independent, identically distributed (i.i.d.) random variables $\{X_n\}_{n=1}^N$, which we call "observations", revealed to us one by one sequentially. There are two possible probability distributions for these observations, with probability density functions (PDF) $f_0(x)$ and $f_1(x)$. If $f_0(x)$ is encountered then we say that the observations correspond to a "unoccupied" channel ($\mathcal{H}_0$), otherwise, the channel is "occupied" ($\mathcal{H}_1$). The decision maker is required to make decision based on the N observations.

In the rest of the paper, we focus on non-negative random variables $\{X_n\}_{n=1}^N$ with general distribution. However, for tractability purposes we only consider exponential random variables in some of the proofs.

III. Conventional Consideration

Since the size of observations is a fixed number N, the optimal statistical test (under various criteria, in particular, the Neyman-Pearson criterion) is the log-likelihood ratio test (LLRT), whose test statistic is

$$T'(\underline{x}) = \sum_{n=1}^{N} \log \frac{f_1(x_n)}{f_0(x_n)}, \tag{1}$$

when the observations are $\underline{x} = [x_1, x_2, \ldots, x_N]$. The corresponding testing rule is a threshold comparison, namely deciding "occupied" if T' is greater than a certain threshold and "unoccupied" otherwise. For example, a typical Neyman-Pearson detector δ' behaves like

$$\delta'(\underline{x}) = \mathcal{H}_1 \quad \text{if } T'(\underline{x}) > \tau'; \quad \text{and } \mathcal{H}_0 \quad \text{otherwise} \tag{2}$$

where the threshold τ' is chosen so that the false alarm probability is a prescribed fixed number $0 < \alpha < 1$, *i.e.*,

$$\mathbf{Pr}_{f_0(\cdot)}\left[\delta'(\underline{X}) = \mathcal{H}_1\right] = \alpha. \tag{3}$$

In the following, we shall denote the false alarm probability and the miss probability achieved by the Neyman-Pearson detector by P_{FA} $(= \alpha)$ and P_{M}, respectively.

Of course, the LLRT decision rule remains optimal under scaling and transformation. For example, consider $f_0(x) = e^{-x}$ and $f_1(x) = 1/(1+\rho)e^{-x/(1+\rho)}$, both defined for $x \geq 0$. Then the LLRT statistic is

$$T'(\underline{x}) = \sum_{n=1}^{N} \left[\frac{\rho}{1+\rho} x_n - \log(1+\rho)\right]. \tag{4}$$

Consequently, the Neyman-Pearson detector can be equivalently rewritten as

$$\delta(\underline{x}) = \mathcal{H}_1 \quad \text{if } T(\underline{x}) > \tau; \quad \text{and } \mathcal{H}_0 \quad \text{otherwise} \tag{5}$$

where

$$T(\underline{x}) = \sum_{n=1}^{N} x_n, \tag{6}$$

$$\text{and} \quad \tau = \frac{1+\rho}{\rho}\left[\tau' + N\log(1+\rho)\right]. \tag{7}$$

IV. Adaptive Detector

In the above example of exponential distributions, we notice that, the test statistic, $T(\underline{x})$, is the accumulation of N nonnegative random variables. Therefore, if we consider the accumulation process as being sequential in time, then whenever a partial accumulation, $\sum_{k=1}^{n} x_k$, $n \in \{1, 2, \ldots, N\}$, exceeds τ, there is no loss of optimality to stop accumulating more observations. So if the LLRT statistic $\log f_1(X)/f_0(X)$ can be transformed into a nonnegative random variable $g(X)$ which possesses a density and satisfies $\mathbf{E}_{f_0}[g(X)] < \mathbf{E}_{f_1}[g(X)]$, we have the following adaptive detector,

Set $T = 0$;
For n from 1 to N
  $T = T + g(X_n)$;
  If $(T > \tau)$ $\quad \delta_{\mathrm{adapt}} = \mathcal{H}_1$; Break and stop.
If $(T \leq \tau)$ $\quad \delta_{\mathrm{adapt}} = \mathcal{H}_0$;

Due to the non-negativity of $g(X)$, $\sum_{k=1}^{n} g(X_k)$ is monotonically non-decreasing with n. Therefore the event $\{\sum_{k=1}^{n} g(X_k) > \tau\}$ for any $n \in \{1, 2, \ldots, N\}$ implies the event $\left\{\sum_{k=1}^{N} g(X_k) > \tau\right\}$, and we have:

$$
\begin{aligned}
& \text{false alarm probability:} \\
P_{\mathrm{FA}} &= \mathbf{Pr}_{f_0}\left[\sum_{k=1}^{N} g(X_k) > \tau\right] \\
&= 1 - \mathbf{Pr}_{f_0}\left[\sum_{k=1}^{N} g(X_k) \leq \tau\right] \\
&= 1 - \mathbf{Pr}_{f_0}\left[\sum_{k=1}^{n} g(X_k) \leq \tau, \forall n = 1, 2, \ldots, N\right] \\
&= 1 - \mathbf{Pr}_{f_0}\left[\delta_{\mathrm{adapt}} = \mathcal{H}_0\right] \\
&= \mathbf{Pr}_{f_0}\left[\delta_{\mathrm{adapt}} = \mathcal{H}_1\right]. \qquad (8) \\
& \text{miss probability:} \\
P_{\mathrm{M}} &= \mathbf{Pr}_{f_1}\left[\sum_{k=1}^{N} g(X_k) \leq \tau\right] \\
&= \mathbf{Pr}_{f_1}\left[\sum_{k=1}^{n} g(X_k) \leq \tau, \forall n = 1, 2, \ldots, N\right] \\
&= \mathbf{Pr}_{f_1}\left[\delta_{\mathrm{adapt}} = \mathcal{H}_0\right]. \qquad (9)
\end{aligned}
$$

That is, the adaptive detector achieves the same false alarm probability and miss probability as the Neyman-Pearson detector

$$
\delta(\underline{x}) = \mathcal{H}_1 \quad \text{if } \sum_{n=1}^{N} g(X_n) > \tau; \quad \text{and } \mathcal{H}_0 \quad \text{otherwise.}
$$

The probability that the adaptive detector stops right after accumulating n observations, for $n = 1, 2, \ldots, N-1$, is

$$
\begin{aligned}
P_n &= \mathbf{Pr}[\sum_{k=1}^{n-1} g(X_k) \leq \tau, \sum_{k=1}^{n} g(X_k) > \tau] \\
&= \mathbf{Pr}[\sum_{k=1}^{n-1} g(X_k) \leq \tau] \\
&\quad - \mathbf{Pr}[\sum_{k=1}^{n-1} g(X_k) \leq \tau, \sum_{k=1}^{n} g(X_k) \leq \tau] \\
&= \mathbf{Pr}[\sum_{k=1}^{n-1} g(X_k) \leq \tau] - \mathbf{Pr}[\sum_{k=1}^{n} g(X_k) \leq \tau]. \qquad (10)
\end{aligned}
$$

Here the probability distribution is either $f_0(x)$ or $f_1(x)$. For convenience, we denote $\mathbf{Pr}_{f_1}[\sum_{k=1}^{n} g(X_k) \leq \tau]$ by β_n, for $n = 0, 1, \ldots, N-1$, where we set $\beta_0 = 1$.

In the adaptive detector, we always decide $\mathcal{H}_0$ after collecting all the N observations, but decide $\mathcal{H}_1$ adaptively based on the sequential realization of these observations. Hence, it is of interest to examine the statistical behavior of the actual number of observations needed to make decision under $\mathcal{H}_1$, and in this paper we focus on its expectation. The expectation of the number of observations under $\mathcal{H}_1$ is

$$
\begin{aligned}
\bar{N} &= \sum_{n=1}^{N-1} nP_n + N\left(1 - \sum_{n=1}^{N-1} P_n\right) \\
&= \sum_{n=1}^{N-1} n(\beta_{n-1} - \beta_n) + N\left(1 - \sum_{n=1}^{N-1} (\beta_{n-1} - \beta_n)\right) \\
&= \sum_{n=0}^{N-1} \beta_n. \qquad (11)
\end{aligned}
$$

V. Example of Expected Number of Observations

Consider $f_0(x) = e^{-x}$ and $f_1(x) = 1/(1+\rho)e^{-x/(1+\rho)}$, both defined for $x \geq 0$. This models that the signal samples are under i.i.d. Rayleigh fading across the observations. From our previous discussion, we choose $g(x) = x$ without loss of optimality. With a threshold τ, we have that the false alarm probability and the miss probability are

$$
\begin{aligned}
P_{\mathrm{FA}} &= 1 - \mathbf{Pr}_{f_0}\left[\sum_{k=1}^{N} X_k \leq \tau\right] \\
&= 1 - \frac{\int_0^\tau t^{N-1} e^{-t} dt}{(N-1)!} \\
&= e^{-\tau} \sum_{k=0}^{N-1} \frac{\tau^k}{k!}; \qquad (12) \\
P_{\mathrm{M}} &= \mathbf{Pr}_{f_1}\left[\sum_{k=1}^{N} X_k \leq \tau\right] \\
&= 1 - e^{-\tau/(1+\rho)} \sum_{k=0}^{N-1} \frac{[\tau/(1+\rho)]^k}{k!}, \qquad (13)
\end{aligned}
$$

respectively. We also have

$$\begin{aligned}\beta_n &= \mathbf{Pr}_{f_1}[\sum_{k=1}^{n} X_k \leq \tau] \\ &= 1 - e^{-\tau/(1+\rho)} \sum_{k=0}^{n-1} \frac{[\tau/(1+\rho)]^k}{k!}, \quad (14)\end{aligned}$$

for $n = 1, 2, \ldots, N-1$. So the mean of the number of observations under $\mathcal{H}_1$ is

$$\begin{aligned}\bar{N} &= \sum_{n=0}^{N-1} \beta_n \\ &= N - e^{-\tau/(1+\rho)} \sum_{n=1}^{N-1} \sum_{k=0}^{n-1} \frac{[\tau/(1+\rho)]^k}{k!} \\ &= N - e^{-\tau/(1+\rho)} \sum_{k=0}^{N-2} (N-1-k) \frac{[\tau/(1+\rho)]^k}{k!}. (15)\end{aligned}$$

It is interesting to capture the benefit of the adaptive detection scheme via the following quantity:

$$\frac{\bar{N}}{N} = 1 - e^{-\tau/(1+\rho)} \sum_{k=0}^{N-2} \left(1 - \frac{k+1}{N}\right) \frac{[\tau/(1+\rho)]^k}{k!}. \quad (16)$$

For large N, it is tempting to perform the following asymptotic approximation: The term

$$\sum_{k=0}^{N-2} \frac{[\tau/(1+\rho)]^k}{k!}$$

is close to $e^{\tau/(1+\rho)}$, and hence the only remaining term in $\bar{N}/N$ is

$$\frac{e^{-\tau/(1+\rho)}}{N} \sum_{k=0}^{N-2} (k+1) \frac{[\tau/(1+\rho)]^k}{k!},$$

which can be further approximated by

$$\begin{aligned}& \frac{e^{-\tau/(1+\rho)}}{N} \sum_{k=0}^{\infty} (k+1) \frac{[\tau/(1+\rho)]^k}{k!} \\ \approx & \frac{e^{-\tau/(1+\rho)}}{N} \left(\frac{\tau}{\rho+1} + 1\right) e^{\tau/(1+\rho)} \\ \approx & \frac{\tau/N}{\rho+1}.\end{aligned}$$

Since $\sum_{n=1}^{N} X_n$ is a chi-square random variable with mean N under $\mathcal{H}_0$ and with mean $(1+\rho)N$ under $\mathcal{H}_1$, for sufficiently large N, in order to achieve small $P_{\rm FA}$ and $P_{\rm M}$, τ/N needs to be within the interval $(1, 1+\rho)$. Therefore, we may parametrize the threshold τ as $\tau = (1+\kappa)N$ where $\kappa \in (0, \rho)$. Consequently, $\bar{N}/N$ may be roughly thought of as $(\kappa+1)/(\rho+1)$, which quickly drops below one as ρ, the signal-to-noise ratio, increases.

As a numerical illustration, consider $\rho = 4$ (*i.e.*, 6 dB), $N = 20$, and fix $\tau = 35$. These parameters lead to $P_{\rm FA} \approx 2 \times 10^{-3}$ and $P_{\rm M} \approx 4.5 \times 10^{-5}$. The expected number of observations can be evaluated as approximately 7.02, so $\bar{N}/N \approx 0.351$, which is highly consistent with $(\tau/N)/(\rho+1) = 0.35$. In Figure 3 we plot $\bar{N}$ versus N for several different choices of ρ and κ. We observe that, even for very small values of N, the linear approximation $\bar{N}/N \approx (\kappa+1)/(\rho+1)$ is still a reasonably accurate estimate for $\bar{N}$.

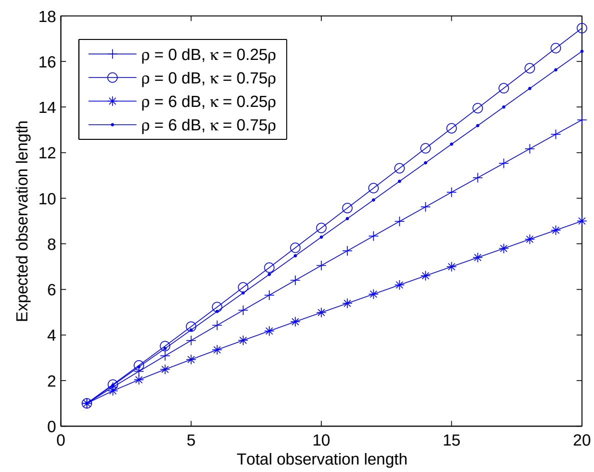

Fig. 3. The expected observation length $\bar{N}$ as a function of N.

We note that, however, the above asymptotic approximation is mathematically flawed. This is because we ignored the fact that besides N, τ also grows unbounded, at the same pace with N. In Appendix A, we present a more rigorous proof to establish this asymptotic approximation.

VI. Comparison with SPRT

The adaptive detection scheme has a hard limit N on the number of observations, and has only one threshold allowing early decision of $\mathcal{H}_1$ but restricting the decision of $\mathcal{H}_0$ toward the end of the N observations. In contrast, the sequential probability ratio test (SPRT) accumulates the LLRT statistics sequentially, with no hard limit on the number of observations. There are two thresholds in SPRT. The detector decides $\mathcal{H}_0$ whenever the accumulated LLRT statistics drop below the lower threshold, and decides $\mathcal{H}_1$ whenever the accumulated LLRT statistics exceed above the upper threshold, but remains undecided if the accumulated LLRT statistics wander between the two thresholds.

It is interesting to compare the expected numbers of observations under $\mathcal{H}_1$ for the adaptive detection scheme and for SPRT. We examine this for the example described in the previous section. We focus on the regime where N grows large, and still parameterize the threshold τ in the adaptive detection scheme by $\tau = (1+\kappa)N$, where $\kappa \in (0, \rho)$. For sufficiently large N, the expected number of observations under $\mathcal{H}_1$ satisfies

$$\frac{\bar{N}_{\rm adapt}}{N} \sim \frac{\kappa+1}{\rho+1}. \quad (17)$$

Now consider the SPRT. According to [1, pp. 108, (Eqn) III.D.23], the expected number of observations under $\mathcal{H}_1$ can be approximated by

$$\begin{aligned}&\bar{N}_{\rm SPRT} \sim \\ &\frac{1}{\mu_1}\left[(1-P_{\rm FA})\log\frac{P_{\rm M}}{1-P_{\rm FA}} + P_{\rm FA}\log\frac{1-P_{\rm M}}{P_{\rm FA}}\right], \quad (18)\end{aligned}$$

where μ_1 is the expectation of $\log[f_1(X)/f_0(X)]$ under $\mathcal{H}_1$. For the example, we have $\mu_1 = \rho - \log(1+\rho)$. As N grows large, for every $\kappa \in (0, \rho)$, both $P_{\rm FA}$ and $P_{\rm M}$ vanish asymptotically. Consequently, we can approximate $\bar{N}_{\rm SPRT}$ by

$$\bar{N}_{\rm SPRT} \sim \frac{-\log P_{\rm FA}}{\rho - \log(1+\rho)}. \tag{19}$$

From Cramer's theorem in the theory of large deviations [2], we can approximate $-\log P_{\rm FA}$ (up to the dominating term) by $N[\kappa - \log(\kappa+1)]$; see Appendix B. Therefore we have

$$\frac{\bar{N}_{\rm SPRT}}{N} \sim \frac{\kappa - \log(\kappa+1)}{\rho - \log(\rho+1)}. \tag{20}$$

A comparison between $\frac{\bar{N}_{\rm adapt}}{N}$ and $\frac{\bar{N}_{\rm SPRT}}{N}$ hence reveals that for sufficiently large N,

$$\frac{\bar{N}_{\rm adapt}}{\bar{N}_{\rm SPRT}} = \left(\frac{\kappa+1}{\kappa - \log(\kappa+1)}\right) \Big/ \left(\frac{\rho+1}{\rho - \log(\rho+1)}\right), \tag{21}$$

which is always greater than one for $\kappa \in (0, \rho)$, implying that the adaptive detection scheme on average requires more observations than SPRT to achieve the same $(P_{\rm FA}, P_{\rm M})$ performance; see Figure 4. This observation is not surprising

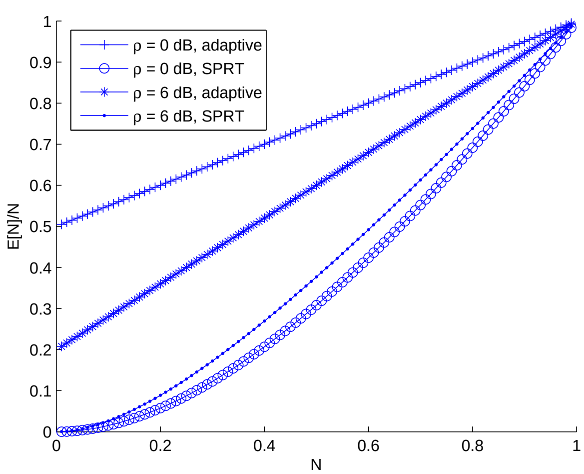

Fig. 4. Comparison of $\bar{N}_{\rm adapt}/N$ and $\bar{N}_{\rm SPRT}/N$.

because according to the Wald-Wolfowitz theorem (see [1, Prop. III.D.1]), SPRT minimizes the expected number of observations among all sequential (thus including fixed-sample-size) statistical tests. We note that, however, such saving on the expected number of observations of SPRT is achieved without a hard limit on the number of observations, and thus occasionally the SPRT requires more than N observations to terminate. In contrast, the decision procedure of the adaptive detection scheme never exceeds N observations.

VII. Minimization of Expected Number of Observations

An open problem is whether the adaptive detection scheme described above minimizes the expected number of observations among all length-N detection procedures that allow early decision of $\mathcal{H}_1$, while achieving the same false alarm probability and miss probability as the fixed length-N Neyman-Pearson detector. To settle this problem, we need to choose a sequence of N thresholds $\{\tau_n\}_{n=1}^N$, and modify the decision rule as: if $T[n] = \sum_{k=1}^n g(X_k) > \tau_n$ for any $1 \le n \le N$, then decide $\mathcal{H}_1$, otherwise decide $\mathcal{H}_0$. Denote the false alarm probability and miss probability of the fixed length-N Neyman-Pearson detector by $P_{\rm FA}$ and $P_{\rm M}$, respectively. We require that,

$$\mathbf{Pr}_{f_0}[T[n] < \tau_n, \forall n = 1, 2, \ldots, N] = 1 - P_{\rm FA}, \tag{22}$$

$$\text{and} \quad \mathbf{Pr}_{f_1}[T[n] < \tau_n, \forall n = 1, 2, \ldots, N] = P_{\rm M}. \tag{23}$$

Denote $\mathbf{Pr}_{f_1}[T[k] < \tau_k, \forall k = 1, \ldots, n]$ by γ_n, for $n = 0, 1, \ldots, N-1$, where we set $\gamma_0 = 1$. Following the same line of derivation as in the adaptive detection scheme, we find that the expected number of observations under $\mathcal{H}_1$ is

$$\bar{N} = \sum_{n=0}^{N-1} \gamma_n. \tag{24}$$

The problem we seek to settle thus becomes: finding $\{\tau_n\}_{n=1}^N$ to minimize $\bar{N}$, while satisfying (22) and (23).

These two constraints, in general, are difficult to satisfy except for the adaptive detection scheme in which $g(x)$ is nonnegative and τ_n is constant across $n = 1, 2, \ldots, N$. Regarding the general case, we have not obtained a negative result; instead, here we consider the example described in Section V and focus on the special case $N = 2$.

With the threshold for the fixed-length Neyman-Pearson detector being τ, we need to choose τ_1 and τ_2 such that the previous two constraints are satisfied. It can be straightforwardly shown that if such τ_1 and τ_2 exist, then they satisfy $0 < \tau_1 \le \tau \le \tau_2$. The false alarm probability constraint is

$$\begin{aligned} &\mathbf{Pr}_{f_0}[X_1 < \tau_1, X_1 + X_2 < \tau_2] = 1 - P_{\rm FA}, \\ i.e., \quad &\int_{x_1=0}^{\tau_2}\int_{x_2=0}^{\tau_2 - x_1} e^{-x_1 - x_2} dx_1 dx_2 = 1 - P_{\rm FA}, \\ &e^{-\tau_1} + \tau_1 e^{-\tau_2} = P_{\rm FA} = (1+\tau)e^{-\tau}. \end{aligned} \tag{25}$$

The miss probability constraint is

$$\begin{aligned} &\mathbf{Pr}_{f_1}[X_1 < \tau_1, X_1 + X_2 < \tau_2] = P_{\rm M}, \\ i.e., \quad &e^{-\tau_1/(1+\rho)} + \tau_1/(1+\rho) e^{-\tau_2/(1+\rho)} = 1 - P_{\rm M} \\ &= [1 + \tau/(1+\rho)]e^{-\tau/(1+\rho)}. \end{aligned} \tag{26}$$

Now, let us consider the relationship between $(x, y) \in [0, +\infty)^2$ described by

$$\begin{aligned} &e^{-x/K} + x/K e^{-y/K} = V(K), \quad \text{that is,} \\ &y = K \log x - K \log\left(V(K) - e^{-x/K}\right) - K \log K, \end{aligned} \tag{27}$$

where $K \ge 1$ and $0 < V(K) = (1 + \tau/K)e^{-\tau/K} < 1$. In order for the relationship (27) to make sense, we need $V(K) - e^{-x/K} > 0$, *i.e.*, $x > -K \log V(K)$. For every K, y decreases with x, from infinity when $x \to -K \log V(K)$, and drops to τ when $x = \tau$. Note that (25) corresponds to $K = 1$, and (26) corresponds to $K = 1 + \rho$.

The derivative dy/dx is

$$\frac{dy}{dx} = \frac{1}{x/K} - \frac{1}{V(K)e^{x/K} - 1}. \tag{28}$$

We can find that for every x, dy/dx is monotonically decreasing as K increases. Therefore, for any two different values of K, the only point on the (x, y)-plane at which the two corresponding $y(x)$ curves intersect is $(x, y) = (\tau, \tau)$; see Figure 5 for an illustration. That is, the only solution that yields

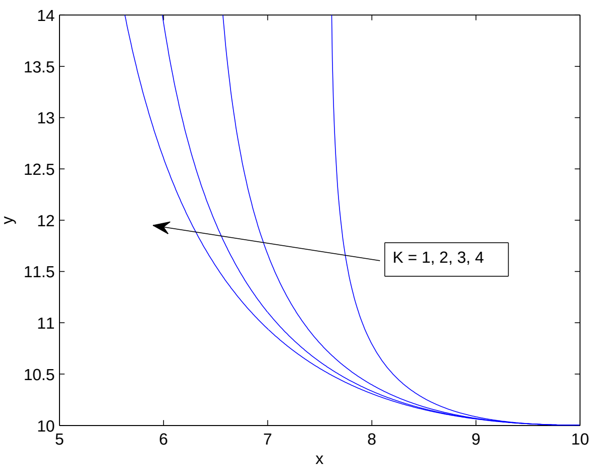

Fig. 5. The relationship $y(x)$ given by (27).

the same $(P_{\rm FA}, P_{\rm M})$ performance as the Neyman-Pearson detector is $\tau_1 = \tau_2 = \tau$. So we conclude that for $N = 2$, there is no other decision procedures that achieve the Neyman-Pearson performance besides the adaptive detection scheme that we have described in Section IV, and that this adaptive detection scheme indeed minimizes the expected number of observations.

VIII. Concluding Remarks

This paper proposed an adaptive spectrum sensing method, which reduces the number of sensing times while maintaining the same detection performance. The key observation is that the decision statistic is an accumulation of non-negative random variables, and hence is non-decreasing. Based on this observation, whenver the decision statistic exceeds the threshold, there is no need to accumulate further observations and a detection can be declared immediately. Expected number of observations associated with the proposed adaptive algorithm is studied, and the connection to the sequential probability ratio test is built.

Appendix

A.

In this appendix, we rigorously prove that in the example of Section V, for $\tau = (1+\kappa)N$ where $\kappa \in (0, \rho)$, as N grows without bound, the ratio $\bar{N}/N$ converges to $(\kappa+1)/(\rho+1)$.

Let us start with

$$\begin{aligned}\frac{\bar{N}}{N} &= 1 - e^{-\tau/(1+\rho)} \sum_{k=0}^{N-2} \left(1 - \frac{k+1}{N}\right) \frac{[\tau/(1+\rho)]^k}{k!} \\ &= 1 - e^{-\tau/(1+\rho)} \sum_{k=0}^{N-2} \frac{[\tau/(1+\rho)]^k}{k!} \\ &\quad + \frac{e^{-\tau/(1+\rho)}}{N} \sum_{k=0}^{N-2} \frac{(k+1)[\tau/(1+\rho)]^k}{k!}. \end{aligned} \tag{29}$$

By noting that $P_{\rm M}$ is given by (13), we have

$$\begin{aligned}\frac{\bar{N}}{N} &= P_{\rm M} + e^{-\tau/(1+\rho)} \frac{[\tau/(1+\rho)]^{N-1}}{(N-1)!} \\ &\quad + \frac{e^{-\tau/(1+\rho)}}{N} \left[1 + \sum_{k=1}^{N-2} \frac{[\tau/(1+\rho)]^k}{(k-1)!} + \sum_{k=1}^{N-2} \frac{[\tau/(1+\rho)]^k}{k!}\right] \\ &= P_{\rm M} + \frac{1}{N} e^{-\tau/(1+\rho)} + \frac{1-P_{\rm M}}{N}\left(1 + \frac{\tau}{1+\rho}\right) \\ &\quad - e^{-\tau/(1+\rho)} \frac{[\tau/(1+\rho)]^N}{N!}, \end{aligned} \tag{30}$$

after collecting terms.

Now as N grows without bound, for every $\kappa \in (0, \rho)$, we have

$$P_{\rm M} \to 0, \quad \text{(weak law of large numbers)} \tag{31}$$

$$\frac{1}{N} e^{-\tau/(1+\rho)} = \frac{1}{N} e^{-\frac{1+\kappa}{1+\rho}N} \to 0 \tag{32}$$

$$\begin{aligned}\frac{1-P_{\rm M}}{N}\left(1 + \frac{\tau}{1+\rho}\right) &= \frac{1-P_{\rm M}}{N}\left(1 + \frac{1+\kappa}{1+\rho}N\right) \\ &= \frac{1-P_{\rm M}}{N} + (1-P_{\rm M})\frac{1+\kappa}{1+\rho} \to \frac{1+\kappa}{1+\rho}, \end{aligned} \tag{33}$$

and from the Stirling's formula,

$$\begin{aligned}& e^{-\tau/(1+\rho)} \frac{[\tau/(1+\rho)]^N}{N!} \\ &= e^{-\frac{1+\kappa}{1+\rho}N} \frac{[N(1+\kappa)/(1+\rho)]^N}{N!} \\ &\approx e^{-\frac{1+\kappa}{1+\rho}N} \frac{[eN(1+\kappa)/(1+\rho)]^N}{N^N\sqrt{2\pi N}} \\ &= e^{-\frac{1+\kappa}{1+\rho}N} \frac{[e(1+\kappa)/(1+\rho)]^N}{\sqrt{2\pi N}} \\ &= \frac{1}{\sqrt{2\pi N}} e^{-\left(\frac{1+\kappa}{1+\rho} - 1 - \log\frac{1+\kappa}{1+\rho}\right)N} \to 0, \end{aligned} \tag{34}$$

because $t - \log(1+t) > 0$ for all $t > -1$.

From the above asymptotics, we thus have $\bar{N}/N \to (1+\kappa)/(1+\rho)$ as N grows without bound.

B.

In this appendix, we prove that $-\log P_{\rm FA}/N$ approaches $\kappa - \log(\kappa+1)$ as N grows without bound. We have from (12),

$$\begin{aligned}P_{\rm FA} &= 1 - \mathbf{Pr}_{f_0}\left[\sum_{k=1}^{N} X_k \le \tau\right] \\ &= \mathbf{Pr}_{f_0}\left[\frac{1}{N}\sum_{k=1}^{N} X_k > 1+\kappa\right]. \end{aligned} \tag{35}$$

So a direct application of Cramer's theorem (see [2, Thm. 2.2.3]) yields

$$\lim_{N\to\infty} \frac{\log P_{\mathrm{FA}}}{N} = \sup_{t>0} \left\{ t(1+\kappa) - \log \mathbf{E}\left[e^{tX}\right] \right\}. \tag{36}$$

For f_0, $\mathbf{E}\left[e^{tX}\right]$ is equal to $1/(1-t)$, and hence we have

$$\lim_{N\to\infty} \frac{\log P_{\mathrm{FA}}}{N} = \sup_{t>0} \left\{ t(1+\kappa) + \log(1-t) \right\}, \tag{37}$$

whose maximizer t^* can be obtained through

$$\frac{d}{dt}\left[t(1+\kappa)+\log(1-t)\right] = 1+\kappa-\frac{1}{1-t} = 0, \qquad i.e., \quad t^* = \frac{\kappa}{1+\kappa}, \tag{38}$$

leading to

$$\begin{aligned}\lim_{N\to\infty} \frac{\log P_{\mathrm{FA}}}{N} &= t^*(1+\kappa)+\log(1-t^*) \\ &= \kappa - \log(\kappa+1). \end{aligned} \tag{39}$$

REFERENCES

[1] H. V. Poor, *An Introduction to Signal Detection and Estimation*, Second Edition, Springer-Verlag: New York, 1994.

[2] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, Springer, New York, 1998.

[3] *Second Report and Order and Memorandum Opinion and Order In the Matter of Unlicensed Operation in the TV Broadcast Bands, Additional Spectrum for Unlicensed Devices Below 900 MHz and in the 3 GHz Band*, Federal Communication Commision, Document 08-260, Nov. 14, 2008.

[4] S. J. Shellhammer, A. K. Sadek, and W. Zhang, *Technical Challenges for Cognitive Radio in the TV White Space Spectrum*, ITA 2009.